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ORIGINAL PAPER

Structures and representations used by 6th graders when working with quadratic functions

Rafael Ramírez[1] · María C. Cañadas[1] · Alba Damián[1]

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Abstract

This study lies within the field of early-age algebraic thinking and focuses on describing the functional thinking exhibited by six sixth-graders (11- to 12-year-olds) enrolled in a curricular enhancement program. To accomplish the goals of this research, the structures the students established and the representations they used to express the generalization of the functional relationship were analyzed. A questionnaire was designed with three geometric tasks involving the use of continuous variables in quadratic functions. The students were asked to calculate the areas of certain figures for which some data were known, and subsequently to formulate the general rule. The results show that the participating students had difficulties expressing structures involving quadratic functions. However, they displayed the potential to use different types of representations to establish the functional relationship. The originality of this study lies in the differences observed in the process of generalization with discrete variables, since, in the case of continuous variables, students could recognize the general expression from analyzing the set of values that can be attributed to the variables in an interval.

Keywords Functional thinking · Geometric formulas · Quadratic functions · Representation · Structure

1 Introduction

A considerable body of research on algebraic thinking in elementary school has been generated over the last three decades. The earliest studies proposed a curricular approach known as early algebra. That proposal sought to promote algebraic thinking, prompt generalization among elementary students and enhance their capacity to express general mathematical relationships (Kaput, 1999).

One of the components of the early algebra approach entails working with functions, i.e., functional thinking. Functional thinking is understood here to be a "component of algebraic reasoning based on the construction, description and representation of and reasoning with and about functions and their constituent elements" (Cañadas & Molina, 2016, p. 3). Analyzing relationships between covarying quantities, generalizing, problem solving, modelling, justifying and predicting are practices associated with algebraic thinking (Kieran, 2004) and functional thinking in particular. Studies focusing on one or several of those ideas can be found in the literature (Brizuela & Earnest, 2008; Cañadas et al., 2016). From the mathematical content viewpoint, such studies are consistently based on linear functions involving discrete variables and natural numbers.

In experimentation with patterns, the connection between algebra and generalization has been emphasized to promote the development of algebraic thinking, but few studies have focused on non-linear patterns (Amit & Neira, 2008). Research focusing on quadratic relationships is in demand due to quadratic relationships' essential role in the study of nonlinear functions (Wilkie, 2022). Given students' recognized difficulties with this type of function (McCallum, 2018) and the possible higher cognitive demand involved in moving from discrete to continuous variables (Stephens et al., 2017), in this study we explore the productions of a group of high-performing students. Our intention in working with this collective is to avoid possible difficulties arising from a poor understanding of the mathematical concepts.

Most authors deem generalization to be pivotal to algebraic thinking (Mason et al., 1985, 2009) contend that, before generalizing, students must identify structure. Kieran (1989) looks upon structure as the system made up of a set of numbers/variables, operations and properties of operations

✉ María C. Cañadas
mconsu@ugr.es

[1] University of Granada, Granada, Spain

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that organize all the elements of an entity in their relationships to each other. The first aim of this exploratory study is to investigate the structures with which students identify the functional quadratic relationship: the expression's component terms, signs, the order of the different elements and the relationships among them. With discrete variables, generalization plays a significant role in how students progress from particular cases to more-distant cases (Hitt & González-Marín, 2016). In contrast, the use of continuous variables allows us to focus the generalization process on a given interval, for which the student identifies the validity 'for any value'. The possibility of using particular cases with numbers different from the natural number set presents a context that favors generalization beyond extending regularity to more-distant cases, as occurs in the discrete case (McEldoon & Rittle-Johnson, 2010). For example, if a segment of length 6 is reduced by a quantity S, the student can obtain the functional relationship of the remaining length as 6-S by recognizing the validity of the general expression when taking S to be 'any' value in the interval [0,6], having previously checked particular cases with decimal numbers. By contrast, if the variable is discrete, such as adding 6 candies to the number of candies a child already has (C), the student can identify the functional relationship $C+6$ by checking that the expression is correct for particular close cases (e.g., for $C=2$, he would have 8 candies) and extending it to far cases. In this case, the validity of the expression for 'any value' implies extending it to any natural number, for very large numbers and even infinite quantities.

The educational usefulness of representations lies in the fact that people working with representations assign meanings to them and understand the mathematical structures involved (Radford, 1998). Accordingly, the second aim of the research describe in this paper is to analyze the representations used by students to express generalization. Rico (2009) identifies representations as "any tool (sign or graphic) that stands in for mathematical ideas and procedures, enabling individual subjects to broach and interact with mathematical knowledge, recording and conveying their understanding of mathematics" (p. 3). According to several authors, the representations most frequently used include natural language, algebraic symbolism, drawings or diagrams, or some combination of the three (Cañadas & Fuentes, 2015; Carraher & Schliemann, 2007; Torres et al., 2022).

In this study we inquired into the structures and representations used by the participants when solving a task in which they were asked to generalize the rules for quadratic functions involving continuous variables. The findings were expected to reveal participants' potential and difficulties, contributing to filling a knowledge gap, in light of the paucity of precedents for high performers of these ages, while providing information of interest for the design of such tasks for ordinary classroom use.

2 Functional thinking, functions, and variables

A number of authors discussed the possible approaches to, or conceptions, components or perspectives of, a structure or system for working on algebra in the classroom (e.g., Blanton & Kaput 2011; Drijvers et al., 2011; Kieran, 2004; Mason et al., 1985). Although none of the authors aspired to establish an exhaustive classification, they do identify categories on the grounds of common classroom algebra content or procedures. Usiskin (1999) identified the following four approaches: (a) generalized arithmetic, (b) problem-solving procedures, (c) relationships between quantities and functions and (d) structures. One of the approaches recognized by several authors, functions, is the one adopted here, namely, the functional approach.

For example, in the USA, various efforts have been made to develop function-based approaches to school algebra as an alternative to the more traditional equation-based approach (Kieran, 2007). Functions are a mathematical concept introduced in secondary education in most countries. Many questions about how to broach functional thinking with elementary students remain unanswered (Schliemann et al., 2012). Where attempted, the idea is not to introduce functions in earlier grades using the same methods as in secondary education. Rather, the intention is to build on the potential of the constructs to develop aptitudes that will help children reason at their present age and at subsequent levels of education.

Functional thinking is an approach recommended for early years, where algebra can be introduced using specific real-life situations involving quantitative relationships (Drijvers et al., 2011; Molina & Cañadas, 2018) noted that "the manner in which functional relationships are worked with in early algebra is based on furthering the perception and generalization of patterns detected in situations involving two covarying variables" (p. 5).

Functional thinking focuses on the relationship between two (or more) co-varying quantities. It concerns the process that leads from the relationship of specific cases to generalizations of that relationship (Smith, 2008, p. 143). Functional thinking is related to understanding the notion of change and how varying one quantity affects the other.

Function, as a mathematical construct, has been variously defined in the literature from different perspectives. According to Usiskin et al., (2003, p. 68), a function is a law that assigns just one element in set B to any given element in set A (where B may or may not be equal to A). Freudenthal

(1983) stresses the phenomenological importance of functions as relationships between one element that varies freely and another that varies within certain bounds (p. 496).

Each representation provides a different characterization of mathematical objects. Algebraic symbolism is commonly used to express generalization from secondary school onwards. Symbolic representation is very useful for synthetically and accurately representing the relationships between indeterminate quantities. Where algebraic symbolism is used, letters represent variables in which the quantities change, while formulas describe inter-quantity relationships (Freudenthal, 1983). In this research, the variables presented to the students were attributes of length and area that could be measured as quantities (Stephens et al., 2017). The geometrical context enables students to construct relationships between quantities. This quantitative reasoning affords a continuous dynamic perspective of covariation, which involves the coordination of continuous change in one quantity with continuous change in another (Johnson, 2012).

Variables may take on values in different number sets. In fact, the most usual definitions involve real numbers. However, in elementary grades (kindergarten and elementary school), the functions involved in generalization tasks concern natural numbers, in the domain and in the codomain. We considered that this would be a limited approach from the mathematical viewpoint, for functions as mathematical content as well as for other topics and areas, including everyday situations, where most variables involved are continuous. This paper's focus is on functions with continuous variables. A continuous variable is one whose possible values all lie within a given range. This nuance is significant for differentiation from the case of discrete variables, where functional relationships generate one-to-one correspondences between elements in the set of the codomain (B) and elements in the set of the domain (A) (Vinner & Dreyfus, 1989), where A and B are sets of natural numbers.

Additionally, the functions considered for working with algebraic thinking at elementary ages are linear ($f(x) = ax + b$), and their constants (a and b) also take values in natural number sets. In this paper, we deal with quadratic functions.

($f(x) = ax^2 + bx + c$) where a, b and c are natural numbers. One context with which primary school students are familiar is geometry. For example, the measurements of lengths, areas and volumes are expressed with continuous variables with which functional relationships can be established.

However, the structures required in quadratic functions, such as 36-S2 to calculate the area of a square from which we have cut off a corner, may require more complexity in the symbolic representation or hierarchy of operations than linear functions. Working with patterns or linear functions can make the student more familiar with the arithmetic operations involved (e.g., ax + b), since this content carries much weight in the curriculum. For example, situations associated with arithmetic problem types (e.g., adding 2 to the independent variable), are more common than those requiring the power of the independent variable.

3 Structures

The term 'structure' is polysemic in mathematics. Structure is one of the main ideas in mathematics (Kieran, 2018). A mathematical structure refers to sets and the relationships among their respective elements. In abstract algebra, structure may mean groups, rings or fields, among other possibilities. In early algebra, a mathematical structure is an arrangement interrelating and organizing a number of mathematical elements (Mulligan et al., 2010). To the extent that structure draws connections and relationships between mathematical concepts and processes, it serves as a means to analyze how students interpret and generalize regularities (Warren et al., 2013; Blanton & Kaput, 2004) presented a definition of algebraic thinking as the "habit of mind that permeates all of mathematics and that involves students' capacity to build, justify, and express conjectures about mathematical structure and relationships" (p. 142). Mason et al., (2009) contended that, before generalizing, students must identify structure. In algebraic thinking, structure is viewed from the standpoint of its close relationship with generalization (Molina & Cañadas, 2018). The ability to see structure is crucial to performing algebraic transformational activity successfully and making sense of algebraic transformations (Kieran, 2018).

From previous contributions about functional thinking drawn from the literature review, we observe that regularities include relationships between the dependent and independent variables found in a function. In this context, structure is equivalent to the regularity or the pattern between the values of two related (dependent and independent) variables (Torres et al., 2021). The structures recognized by elementary students when generalizing linear functions have been described in the literature. Torres et al., (2022) explored the functional thinking of three second-graders (7- to 8-year-olds) in tasks involving linear functions. For example, in questions about particular cases in a task involving the function $f(x) = x + 3$,, all the students added 3 to the value of the independent variable to get the value of the dependent variable. So, the structure 'seen' by them was x+3. The authors highlighted the scant variety in the structures distinguished by students for a given regularity, on occasion because they identified the correct structure which they then used in their replies to all the items on the questionnaire. The authors also noted that students found it difficult to establish structures involving a combination of operations.

Studies (e.g., Pinto & Cañadas 2018; 2019) with elementary students of different grades revealed students' ability to recognize linear relationships and to generalize more than one equivalent structure (thanks to which, according to the authors, the students' thought processes when generalizing could be determined). In linear functions with discrete variables, the particular cases used in the students' replies enabled them to recognize the structures involved. Literature on the recognition of quadratic structures in continuous functions in elementary grades is scarce; most studies focused on patterns. For example, Amit and Neira (2008) determined that mathematically talented sixth- and seventh-year (11- to 13-year-old) students were able to generalize quadratic patterns. El Mouhayar & Jurdak (2015) focused on studying how the use of strategies to generalize quadratic figural patterns from immediate-near cases to far-n cases varied in students' work across grades 4–11 (ages 9–17).

However, a number of studies (e.g., Ellis 2011a>, Ellis 2011b) have shown that middle school students can generalize quadratic relationships when dealing with the relevant quantities of height, width and area, and then extend this reasoning to develop a correspondence rule. In tasks in geometric contexts in middle grades with quadratic pattern figures, too, the role of visual structures for identifying the functional relationship has been highlighted (Ellis, 2011a; Rivera, 2010; Wilkie, 2022).

Most of the literature on functional thinking concerns work done by mainstream students. However, we find an exception in the study of Wilkie (2021). This author worked with high-achieving secondary students (grades 7–12) who were able to generalize quadratic figural patterns by arranging them with geometric figures. These results provided evidence for the potential of visualization to support algebraic and functional reasoning with quadratic functions (Wilkie, 2022).

4 Representation

Effective mathematical thinking entails understanding the relationships among different ways to represent the same idea, and the structural similarities and differences among them (Goldin & Shteingold, 2001). Our focus in this paper is on what Castro & Castro (1997) call external representation. The co-existence of various types of representation in school algebra helps students visualize abstract mathematical objects. Yerushalmy & Schwartz (1993) acknowledged different types of representations. Each type, including verbal language, algebraic symbolism, numerical symbolism, and tabular, graphic, pictorial and manipulative representation, highlights certain features of mathematical content and their inter-relationships.

A number of authors (e.g., Brizuela & Earnest 2008; Confrey & Smith, 1991) have addressed the idea of combining several types of representation, a strategy that (a) establishes the interconnections between representations, (b) enables students to externalize and visualize different but complementary perspectives of an idea, (c) creates an atmosphere favorable to abstraction and the understanding of mathematical ideas and (d) is a relatively unexplored area in elementary education.

A number of studies on functional thinking address the types of representation used by elementary students to solve generalization problems. Cañadas & Fuentes (2015) described the types of representation used by 32 first-graders (6- to 7-year-olds). The students used pictorial representation most frequently, except in a question involving generalization, where verbal representation prevailed. They also applied numerical and multiple representation (combining pictorial and numerical-symbolic representation).

Other researchers have also used tasks involving linear functions in elementary education. They have highlighted students' preference for expressing generalization verbally (Merino et al., 2013; Torres et al., 2022), and there is evidence that fifth-graders (10- to 11-year-olds) have the potential to use algebraic notation and verbal representation to generalize (Pinto & Cañadas, 2018). When working with particular cases, these students also used pictorial representation, primarily because it was included in the problem wording, but also because it proved to be helpful for calculating certain particular cases. With sixth- to eighth-graders (10- to 15-year-olds) familiar with tasks involving patterns, Akkan (2013) found that sixth-graders were inclined primarily to use numerical representation. In contrast, more-highly performing seventh- and eighth-graders generalized using a variety of representations, including algebraic symbolism. Also, in research involving 33 sixth-graders solving a problem with area calculations, students were able to generalize verbally as well as symbolically, showing a clear preference for a functional approach (Ureña et al., 2022). Moving up a level, middle-school students can represent relationships with variable notation including linear relationships and quadratic relationships (e.g., Ellis 2011a; Francisco & Hähkiöniemi, 2012).

In this paper we describe an experiment in late primary education and aim to investigate geometric contexts of measurement involving continuous variables and quadratic functional relationships.

5 Research objectives

In light of the scarcity of literature on functional thinking in upper primary school students, with regard to continuous variables and quadratic functions, we designed this study with high-achieving students. The study particularly focused on describing the functional thinking exhibited by sixth-grade students participating in a curricular enrichment program. That general objective was specified in terms of two specific objectives, as follows:

SO1: to describe the structures established by high-achieving elementary students when working with a quadratic functional relationship in a geometric problem involving continuous variables;
SO2: to identify and describe the representations used to express the generalization of the functional relationship.

6 Methodology

The qualitative research did not "intend to intrinsically generalize the findings to larger universes, nor necessarily to obtain representative samples, but is based on induction" (Hernández et al., 2010, p. 16). Rather, we sought to discover and describe developments on the grounds of the information collected. This is consequently a descriptive study that also involves exploratory research, given that the functional thinking 6th-graders use when solving quadratic functions has been scantily analyzed in the literature.

We designed and implemented a teaching experiment (Confrey & Lachance, 2000). In the first session, with the whole group, we introduced a task involving a quadratic function with various questions, and the students filled in a questionnaire about the task. Finally, there was a sharing session in which the teacher-researchers interacted with the students, encouraging them to justify their answers and compare them with their classmates'. After this, the students were reminded of some of the concepts concerning calculation of the area of a square and powers as products of equal factors. We carried out an analysis, and a week later we conducted a second session of individual semi-structured interviews in which more-complex tasks were presented. We describe the tasks and the questions below.

6.1 Subjects

We worked with an intentional sample consisting of six sixth-graders (11- to 12-year-olds) attending a charter school in Granada, Spain. The students were enrolled in a curricular enrichment program, which added new content or ideas not included in the formal curriculum or that worked with curricular content in greater depth. These programs supplement ordinary curricular content with tasks designed to develop characteristics associated with mathematical talent, such as originality in problem solving and understanding of abstract ideas deemed complex for students' age (Piñeiro et al., 2017).

The six students had been nominated by their teachers for high-performer status. They were therefore initially assumed to have an interest in improving performance and to possess a sufficient understanding of basic mathematical ideas. Their assessments identified a range of talents. S1 exhibited complex academic talent; S2, artistic and verbal talent; S3, mathematical talent; S4, academic and mathematical talent; S5, verbal talent. While S6's teachers had perceived above-average talent, the student had yet to be assessed by the department. A code consisting of 'S' followed by a number was assigned to each student randomly to ensure anonymity.

The students had not received previous instruction on functional relationships or variables. Their teachers confirmed that they knew about the use of letters to represent the area of a square, but they had not previously worked with generalization tasks or those entailing algebraic thinking.

6.2 Data collection

The tool used to collect information was a questionnaire administered over two consecutive sessions. Participants were given three generalization tasks involving quadratic functions around which a series of questions was asked. Based on our background research and in light of the objectives of this study, we designed a geometric word problem based on three non-linear functions: $A(S)=36\text{-}S^2$, $A(S)=36\text{-}4S^2$ and $A(X,Y)=36\text{-}X^2\text{-}Y^2$.

The information-gathering techniques applied included participant observation, semi-structured interviews, and the students' written answers. Two teacher-researchers participated in both sessions. One was an enrichment program teacher, and the other videotaped the interviews and recorded the observations. The students' usual classroom teacher was also present but did not intervene.

We designed the tasks according to the recommendations for introducing students to functional situations through quantitative reasoning (Ellis, 2011b). We introduced functional relationships through quantitative situations that represented precise data, included quantitative situations with quantities covarying continuously rather than only discretely, and that supported sustained student attention in a given context or situation. The tasks chosen for the questionnaire called for generalizing quadratic functions involving continuous variables. The task required students to find the relationship between the length of the side of a square and the area of the resulting figure; this task involved establishing a continuous quadratic function, since the possible values were confined within an interval.

Many functional relationships deriving from geometric problems, such as the application of length in measurement problems, entail continuous variables. The elementary school curriculum in fact specifically includes models for problems associated with area calculation that require the application of quadratic functional relationships. As discussed below, the task entailed calculating the area of a square based on the length of one of its sides. The two sessions during which the information was collected are described below.

6.3 Session 1

We introduced students to the geometric context of the task in the first session during one hour. A smaller square in the upper right corner was removed from a square with sides measuring 6. We asked the students to determine the relationship between the area of the resulting figure and the side of the corner removed, which was $A(S) = 36\text{-}S^2$, where A(S) was the area of the resulting figure and *S* the side of the corner removed. Both area and (side) length were continuous variables within a range of values delimited by the conditions described in the problem, without mentioning any unit of measurement.

The students worked individually on a questionnaire, followed by a discussion addressing the difficulties they had encountered and showing how to calculate the area of a square and how to use powers to represent it.

The first session entailed answering a total of 13 items (task 1). The wording of the session 1 items is set out below, along with a brief explanation of the ideas addressed in each.

In items 1 and 2, we introduced a square having a side of length 6 and asked students to calculate the square's area.

1. Draw a square with sides = 6
2. Find the area of the square

In items 3, 4, 5 and 6, the task involved removing two squares, first a square having a side measuring 2, and then a square having a side measuring 4, from the upper right corner of the square whose side measures 6. We asked the students to draw the figure after removing the corner square, and to determine the area of the resulting figure.

3. Find the area of a little square with sides that measure 2.
4. Draw a little square with sides = 2 inside the upper right corner of the square shown below. Suppose we remove the small square with sides = 2 from the square with sides = 6. Find the area of the new figure. (See Fig. 1.)

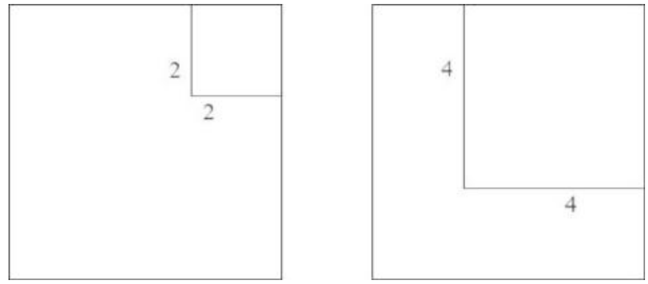

Fig. 1 Figures described in session 1

5. Now find the area of a little square with sides that measure 4.
6. Draw a little square with sides = 4 inside the upper right corner of the square shown below. Suppose we remove the small square with sides = 4 from the square with sides = 6. Find the area of the new figure.

In items 7 through 13, we worked with a square with sides = 6 from whose upper right corner a square of unknown side length was removed (Fig. 1). We asked the students to draw the figure as it would look after removing the smaller square in the corner and find the remaining area.

7. Draw a small square in the upper right corner. How would you explain to a friend how to find the area of a square when you don't know the size of its sides?
8. Now suppose that the little square has sides = *S*. Could you find the area now?
9. Suppose we remove the small square with sides = *S* from the square with sides = 6. Find the area of the new Fig.
10. Do you think the answer could be in decimals? What about negative numbers?
11. Now imagine the smallest possible square inside the larger square. How much would its sides measure? What about the largest one possible? How much would its sides measure?
12. How much would the smallest possible new figure measure? How much would the largest possible new figure measure?
13. How small and how large do you think *S* could be?

6.4 Session 2

The second session consisted of an individual interview with each student conducted by two team researchers, one acting as interviewer and the other taping the interview and recording the observations. The researchers conducted

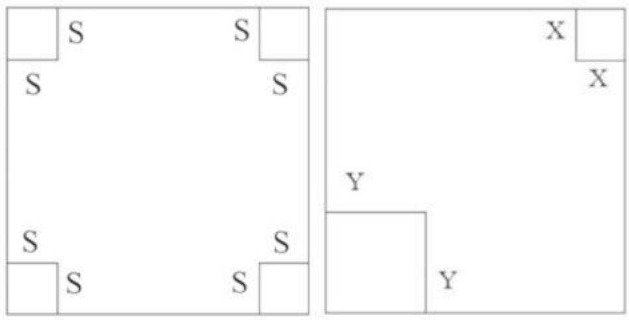

Fig. 2 Figures described in session 2

the individual interviews one week after the first session, and there was no instruction on this topic between the two sessions. The interviews lasted one-half hour each, which proved ample for the work requested of the students. The interviewer verbally asked the same items as appeared in the questionnaire (items 14 to 25), checking that the students understood the information and encouraging them to explain and justify their answers. The items addressed during the interview are listed below, along with a brief explanation of what was sought in each.

In items 14 to 18 (task 2), we worked with a square with sides = 6, and we removed a square of the same unknown area from each corner. We asked the students to draw the squares and find the area of the figure resulting from removing the corner squares (Fig. 2, left). In these items (Fig. 2, left), the function implied was $A(S) = 36-4S^2$, where A(S) was the area of the resulting figure and *S* was the side of the corners removed.

Students were supposed to establish the relationship between the length of the side of the corner squares and the figure resulting from removing the four corners. We also asked about extreme values of the length of the side ($S\_min = 0$, $S\_max = 3$) and the extreme values of the resulting area ($A\_min = 0$, $A\_max = 36$) and determined in both cases whether the students established a range of values or focused on specific values only.

14. Now draw a small square with sides = *S* inside all four corners of a square with sides = 6.
15. Imagine you remove all four small squares. Find the area of the new Fig.
16. Do you think the answer could be in decimals? What about negative numbers?
17. Now imagine the smallest possible square inside the larger square. How much would its sides measure? What about the largest one possible? How much would its sides measure?
18. What values could *S* have in this case?

In items 19 to 25 (task 3), we worked with a square with sides = 6, from which two smaller squares were removed from opposite corners. Their size was unknown, although the two were not necessarily the same size. We asked the students to draw the squares and find the area of the figure resulting from removing the corner squares (Fig. 2, right). Here the relationship was $A(X,Y) = 36-X^2-Y^2$ where A(X,Y) was the area of the resulting figure, *X* was the length of the side of one of the removed corners and *Y* was the length of the side of the other removed corner.

Students were asked to establish the relationship between the sides of the corner squares and the figure resulting from removing the two opposite corners. We analyzed the type of representation used and whether or not students used more than one type. We also recorded whether they used different variables to symbolize each corner.

In these items, we asked the students whether *S* (side) could be a decimal or a negative number. We also asked about its extreme values and the extreme values of the resulting area ($A\_min = 0$, $A\_max = 36$) and determined whether the students established a range of values for both or focused on specific values only.

19. Now draw two small squares inside opposite corners of a square with sides = 6.
20. If we know the two small squares are different, how could we symbolize the value of their sides?
21. What would the area of the new figure be if we remove the two little squares?
22. Do you think the answer could be in decimals? What about negative numbers?
23. Now imagine the smallest possible small square. How much would its sides measure? And the other square?
24. And now the largest possible small square. How much would its sides measure? And the other square?
25. How small and how large could the sides of one of the squares be? And the other one?

The information was analyzed from students' written answers to the above items and the transcription of the session videos.

6.5 Analysis categories

The units analyzed were the students' answers for each item. We established two categories a priori to reply to the two research objectives.

- Structure. Here we determined whether or not students recognized the structure involved. They were deemed to recognize the structure when they mentioned it for the general case or used it to calculate more than one

Table 1 Summary of structures recognized

	Session 1 (items 7–13)		Session 2 (items 14–25)	
Student	Individual answer	Pooled answer	Items 14 to 18	Items 19 to 25
S1	None established	-	None established	$36 - \mathrm{X} = 32$ $36 - \mathrm{Y} = 30$
S2	$36 - \mathrm{S}$	$36 - \mathrm{S}$	$36 - 4S^2$	$36 - X^2 - Y^2$
S3	None established	None established	$36 - 4S^2$	$36 - X^2 - Y^2$
S4	None established	-	$36 - \mathrm{S}$	$36 - X^2 - Y^2$
S5	$36 - \mathrm{S}$	$36 - \mathrm{S}$	$36 - 4\mathrm{S}$	$36 - \mathrm{X} - \mathrm{Y}$
S6	$36 - \mathrm{S}$	$36 - \mathrm{S}$	$36 - 4S^2$	$36 - X^2 - Y^2$

Note. - = the student failed to reply.

particular case. Any structures recognized were defined and represented symbolically by the authors.

- Representation. Here we described the type or types of representation used and classified them as verbal, pictorial, symbolic or multiple. Where multiple representation was used, the types combined were specified in the results.

Verbal language refers to natural oral and natural written language. The former includes a specialized sub-language with oral characteristics related to mathematical domains. Natural written language involves writing out sentences and phrases.

In algebraic symbolism, letters represent variables whose meanings are indeterminate changing quantities, while algebraic expressions describe the relationships among those quantities. Symbolic representation is highly useful to represent synthetically and precisely relationships between indeterminate quantities.

Pictorial representation comprises drawings that express relationships. Its use reduces the cognitive weight of the task for students by helping them visualize and think through the ideas implicit in certain information (National Council of Teachers of Mathematics, 2000).

Multiple representation is the result of combining two or more of the aforementioned types of representation (Van Someren et al., 1998).

7 Analysis of data, and results

Based on the a-priori categories, two of the researchers analyzed the data, classifying structures and representations. Expert triangulation was carried out with a third researcher. Non-matching cases served to determine definitive criteria and confirmed that the categories were complete for the analysis of all responses, with no unstated categories added. For example, although other research focuses on tabular representation (e.g., Blanton & Kaput 2011), we did not consider tabular representation because it was not found in our preliminary data analysis.

The data as analyzed and the findings for each variable are discussed below. The answers to items 1 through 6 are not analyzed, because they were used to introduce the task and determine whether the students knew how to calculate area and the procedure for removing a square from one of the corners in a specific case.

7.1 Structure

The structures recognized by the students are listed in Table 1. In session 1, we distinguished between students' individual work and the discussion for items 7 through 13. In session 2 (interview), we distinguished between the two sets of items.

Three of the six students (S1, S3 and S4) assigned the independent variable specific values and established no structure when working individually in session 1. The other three established the structure incorrectly as $36 - \mathrm{S}$. None identified structures when working with particular cases, although they did for the general case. Despite being high achievers, the students showed by their answers that they were unfamiliar with the use of quadratic expressions. We did not find structures derived from multiplication or powers, such as S^2 or SxS.

Five students (S2, S3, S4, S5, S6) participated in the pooling of ideas. The exception was S1, who attended but did not participate. As a rule, the five recognized the existence of a relationship between the two variables, for they claimed they would need to know the size of the side of the corner square to calculate the area of the larger square and therefore the area of the resulting figure. Students showed no difficulty in using general expressions in the functional relationships derived from considering some particular cases, although they did not express the relationships correctly, as can be observed in the following example of S2: "You can put the L, a number, 3.5" (…) "all not going beyond 0 and 6" (…) "I have put it assuming that I don't have a ruler. I

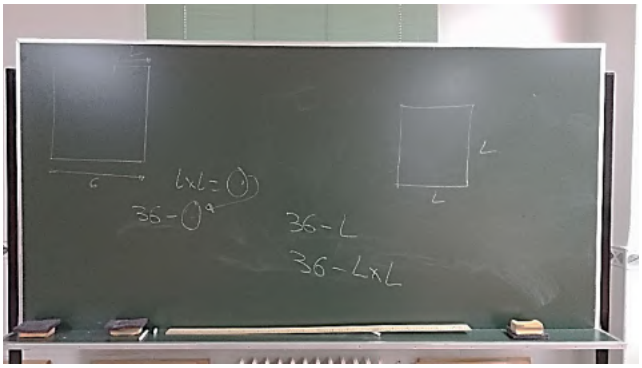

Fig. 3 Pictorial and symbolic representations used by the researchers

said to calculate the area you have to multiply the height and the base of the small square, then the area of the big square. Then, you have to subtract the result of the big square from the small square and that is the result of the area".

One of the researchers guided the collective session for items 7 through 13 by posing questions concerning students' remarks and replies: "Let's talk about this for a minute. What do you think about your classmate's reply? Do you agree?" "How could we go about finding the area of the square?" S6, S2 and S5 established the structure as 36-*S*, and S3 continued to use specific values but gave no sign of recognizing the structure. The researchers prompted the students to generalize symbolically step by step, verifying that all ultimately understood the process and were able to generalize on their own. To do so, the researchers drew pictorial and symbolic representations on the classroom blackboard (Fig. 3) while discussing the answers with the students and guiding the collective exercise. Although all six students ultimately recognized the structure correctly, three (S2, S3 and S6) discovered unaided that the smaller area was *S*×*S* (although they defined the area of the resulting structure incorrectly as 36-*S*).

One of the students (not identified because it was during the collective discussion) used the expression "double *S*" to represent the area of the small square. S2 performed the calculation correctly, but did not symbolize it algebraically, saying: "What I did, instead of representing it like that, *S* times *S* equals, I drew a circle and subtracted it."

The results of the first session show that students identified the need to subtract the area of the square from the total area, but they were not familiar with either the area formula or quadratic expressions. For example, they confused SxS and 2 S or S. They did not consider whether the expression they proposed was fulfilled in the particular cases, which they did calculate correctly. Therefore, they did not perceive the symbolic expression as a generalization of what applied in the particular cases. That is to say, they did not perceive S as a variable that takes values between 0 and 6, even though they answered correctly for the case S = 2. After the discussion, we reminded them that the area of the square can be calculated as S^2 or SxS.

Some students assimilated these concepts, as shown at the beginning of the second session. We found two different kinds of answers in session 2 (items 14 to 18). S2, S3 and S6 gave the correct answer, 36-4S^2, treating the variables as continuous by recognizing the validity for all values of the interval. On the other hand, three of the students continued reporting the same errors as in the previous session. S1 defined the structure while working with particular cases, whereas S4 and S5 identified the structure incorrectly as 36-4 *S* and 36-*S*, respectively. S4 found 36-*S*, because she[1] was unable to correctly express *S*×*S*, despite noting that the area of the small square would be calculated that way; she also failed to include the number 4 in the algebraic formula. S5 included the multiplier (4) but expressed the structure incorrectly as 36-4 *S*, for he found it difficult to arrange the terms and failed to realize that the area in this case would be *S*×*S*.

All the students except S1 correctly used the same letter to designate all four corners, proof that they realized they were equal. All six noted that the area of the small square would have to be multiplied times 4, although S2 required help to do so when the interviewer asked, "Are all the results the same or are they different?" "The area of the small square has to be multiplied how many times?" It is evident that they were familiar with the line structures associated with repeated additions (4 S = S + S + S + S) and even the use of the hierarchy of operations when combining additive and multiplicative structures (seeing no need to use brackets in expressions such as 36-4S^2). They also recognized that the same letter represents equal quantities, in this case the length of the side of the little squares.

In items 19 to 25, four students (S2, S3, S4 and S6) recognized the structure correctly. The other two (S1 and S5) established incorrect structures (respectively, $36 - \mathrm{X} - \mathrm{Y}, 36 - \mathrm{X} = 32, 36 - \mathrm{Y} = 30$). The students persistently confused linear and quadratic variables. In other words, the students prioritized the difference in areas in the structure, but they interpreted both the length of the side and the area of the square with the same letter. They did not identify the quadratic relationship established between the length variable and the area variable. The fact that this difficulty persisted in some high-performing students, despite the instruction received in the first session, makes it an indicator of the complexity of the proposed task for students of these ages without previous instruction in this content.

S2 correctly established the structure verbally in the following terms: "I would need to calculate the area of this square, calculate *S* times *S* and *X* times *X*. I then add the

[1] In this paper, pronoun gender is assigned arbitrarily and randomly to ensure student anonymity.

Table 2 Summary of representations used

	Session 1 (items 7–13)		Session 2 (items 14–25)	
Student	Individual answer	Pooled answer	Items 14 to 18	Items 19 to 25
S1	None established	-	V	S
S2	M (V and P)	M (V, S and P)	M (V and P)	M (V and S)
S3	None established	V	M (V and S)	V
S4	None established	V	S	S
S5	V	M (V and S)	S	M (V and S)
S6	V	M (V and S)	M (V and S)	S

Note. - = no answer; V = verbal; P = pictorial; S = symbolic; M = multiple.

values of *X* times *X* and *S* times *S*, and I subtract that sum from the area of the large square." Both she and S3 defined it symbolically. S3 verbalized the structure correctly by saying, "We multiply *S* times *S*, and then we multiply *m* times *m;* I think then we need to add the two up and subtract it from 36," but used the wrong symbolic expression. All the students used different letters to represent the two corners. Four of them needed prompting, for they reasoned with arguments such as, "If they weren't the same, well, I'd use S for one and a higher number or letter for the other, multiplying *S* times *S*," or "I'd call them the same, because after all, since I do the same thing, you can use the same letter."

7.2 Representation

The representations used by the students are summarized in Table 2 and described in greater detail below.

The three students who replied individually to the session 1 items, S2, S5 and S6, used verbal representation to establish the structure. In the pooled exercise in session 1, all the participating students (S2, S3, S4, S5, S6) used verbal representation, and S5 and S6 used multiple (verbal plus symbolic) representation. S2 combined three types of representation (verbal, symbolic and pictorial).

In the interviews, students used primarily symbolic and verbal representation, separately or by combining the two types together or combining both types with other types. In session 2, three students (S4, S5 and S6) used symbolic representation in both parts of the session, four (S1, S2, S3 and S6) used verbal representation in items 14 to 18, and three (S2, S3 and S5) in items 19 to 25 used also verbal representation. Student S2 used multiple representations in items 14 to 25, while S2, S3 and S6 used multiple representations in the "four corners" part of the problem and S2 and S5 used multiple representations for the "two corners" task. The two combinations used by these students were verbal-symbolic and verbal-pictorial.

In the symbolic representations, S2 used the power factor, although incorrectly, indicating S^8, based on the fact that there were eight sides (eight *S*'s). S3 also raised the variable to a power, but also expressed the terms incorrectly, writing the structure as "6 times 6. And now *S* times *S*…, 36-S^8."

The results show the students' potential to express generalization through multiple representations, mainly verbal and symbolic. Their versatility in showing different representations is proof of the generalization process, despite the errors. For example, despite the difficulties and errors displayed in the quadratic expressions, the students overcame these difficulties in the verbal representations (for example, replacing "SxS" with "the area of the little square").

All the students recognized that X can take all positive values in the interval [0,6], which for them corresponded to decimal numbers. The students demonstrated that the variables associated with measurements help give meaning to decimal expressions and help one speak of the indeterminate in terms of any measurement, such as any value in an interval.

S1: *It can take the values 3.7, 2, 4.4, 5…. No negatives, because it is a measurement.*

S2: *S can take decimal values. You can put in a number, 3.5.*

A difference was found in this kind of function with continuous variables. While in discrete variables (e.g., Torres et al., 2021), generalization is perceived when the student identifies regularity in particular cases and extends it to the rest of the cases involving large numbers (McEldoon & Ritte-Johnson, 2010), in continuous variables generalization is demonstrated when a student recognizes regularity in particular cases and extends it to all the types of numbers he/she identifies in a particular interval.

8 Discussion and conclusions

This study focused on two ideas associated with functional thinking, structure and representation, in working with high-performing sixth-graders, exploring aspects that had not yet been thoroughly studied, such as continuous quadratic functions in geometry contexts. Three main contributions can be drawn from the results, particularly in comparison with previous studies using discrete variable functions.

The participating students had difficulties expressing structures involving S^2.

Although none of the students had received prior instruction on the use of letters, some identified a structure implicit in the first task, establishing $S \times S$ as the area of the corner removed. We believe that to be a significant achievement, inasmuch as it can thus be inferred that they both generalized and recognized structure. When subtracting that sum from 36 (the area of the original square), however, they failed to use the formula for area, identifying the structure for the area of the new figure as 36-*S*. Some students included the power factor in their notation, but incorrectly. That may be attributable to the fact that, while they knew that the area of a square is found by raising the length of its side (when given as a number) to the power of 2, they were unfamiliar with letter-based algebraic notation and possibly thought that the factor differed depending on whether the dimension was expressed in letters or numbers. The error may have also been due to an insufficient prior understanding of the power factor, however.

In the second task, four students recognized that the same letter should be used for all four corners (which were equal) and used it correctly in the established structure. Their error, answering *4 S* instead of $4S^2$ (as the value to be subtracted to find the area of the figure resulting from removal of the four corners), may have been induced by the fact that, although they realized they had to subtract four times something dependent upon *S*, they failed to distinguish whether *S* was the side of the smaller square or the area of the corner removed.

In the third task (items 19 to 25), four of the students recognized the structure as involving a quadratic expression, and four (of the five) also used different letters to designate each corner. That attested to their realization that different corners required different letters. Another interesting finding was the way S4 expressed the terms: instead of $36 - X^2 - Y^2$, she used other letters ($m \times m = n$, $S \times S = a$), and ultimately wrote the relationship as $36 - \mathrm{n} - \mathrm{a}$.

The participating students exhibited difficulties and errors in performing the tasks involving quadratic relationships, as found in previous studies (McCallum, 2018). However, the use of measurement contexts with continuous functions shows that even high-achieving students had difficulties that would not have become manifest either in previous pattern-based studies where students proved able to generalize quadratic patterns (Amit & Neira, 2008; El Mouhayar & Jurdak 2015) or in studies with geometric tasks for students in higher grades (Ellis, 2011a, b; Wilkie, 2022). The observed difficulties in identifying the structure of quadratic relationships stemmed from a poor understanding of the concepts involved (area and length). Despite instruction in area calculation and power notation, errors persisted in some of the students. Although quantitatively rich situations offer a useful context, it is important to provide instructional settings that engage students to focus on or generalize relationships that are accurate, powerful, or even algebraically meaningful to them (Stephens et al., 2017). That finding has significant implications for the design of tasks suitable for such students and the identification of the difficulties their typically developing peers may encounter.

To express generalization, the participating students used varied representations, most frequently verbal and algebraic symbolism.

Moving on to the second research objective, we found that the students used primarily verbal and symbolic representation. While they did not refuse to apply symbolic representation, they did make mistakes. Even in the cases in which they failed to express adequately the correct quadratic expressions, they did not show difficulties in using general expressions from the particular cases they analyzed. Our results on representation agree with the findings reported by Pinto & Cañadas (2018) to the effect that students generalized but used different structures to do so. That induces us to believe that the difficulty lay not in the quadratic function per se, but in expressing the function with symbolic notation.

Of the types of representation initially defined (verbal, symbolic, tabular, pictorial and multiple), students used four, the exception being tabular. Pictorial representation was only scantily deployed, despite its presence in the task wording and introduction. Multiple representation was present as the combination of verbal and pictorial, verbal and symbolic and even verbal, symbolic and pictorial. This study, like others, shows that high-performing students of these ages have the potential to use different representations and to handle algebraic symbolism (Akkan, 2013; Ureña et al., 2022). These findings are consistent with results reported by Cañadas & Fuentes (2015), who observed that verbal and symbolic representation are the two types most frequently used, along with a combination of these two and even all three types.

The present findings are also consistent with Merino et al.'s (2013) observation that verbal is the type of representation most frequently used by students. However, this study has shown the usefulness of combining verbal representation with other types, such as symbolic representation, as a means of externalizing complementary perspectives of an idea (Brizuela & Earnest, 2008; Confrey & Smith, 1991), in this case, to overcome the stated difficulties in expressing the quadratic expressions. The choice of verbal representation by these subjects and those of Merino et al., (2013) might be conjectured to be attributable to the greater ease of explaining their answers verbally than in any of the other forms of representation.

We observed differences in the process of generalization with continuous and discrete variables.

The students showed difficulties perceiving S as a variable, as they did not recognize that the proposed functional relationship was not fulfilled in the particular cases analyzed. However, all the participating students showed no difficulties in recognizing the validity of the expressions used in decimal numbers belonging to a given set. This nuance denotes a substantial difference in the inductive reasoning process with respect to discrete variables, where generalization is perceived when the student identifies regularity in particular cases and extends it to the rest of the cases involving large numbers (McEldoon & Ritte-Johnson, 2010; Torres et al., 2021). In geometric contexts, which use continuous variables associated with measurement, generalization can be reached as the validity of the expression for all the values in an interval (Ellis, 2011b).

In addition to these three contributions, some implications can be drawn for the design of tasks in the latter years of primary school that favor the development of functional thinking. There is some controversy in the research literature over whether the use of real-world or realistic situations favors algebraically useful generalizations (Stephens et al., 2017). Nevertheless, it has been found that, even in high-achieving students, a significant understanding of the concepts associated with the variables used (length, area and their relationships in our case) is necessary. In this type of task, it is important for the student to establish a connection between geometry and algebraic thinking (Cheah et al., 2017). The difficulties encountered lead to different actions to be taken into account in the teaching process associated with functional thinking.

The fact that that the students could recognize the structure by expressing it verbally but not symbolically may place the focus on the use of multiple representations, so that the student can use the representation that he/she finds most understandable. However, the potential demonstrated in the use of symbolic and verbal language to express generalization suggests that these approaches could be transferred to mainstream students, with the appropriate adaptation of task complexity. For example, a priori, the structure for the subtraction of two magnitudes gradually increases in complexity, from finding the length of the side of the resulting Fig. (6-S), to finding the area of the figure after removing (a square in) one corner ($36\text{-}S^2$), removing four identical corners ($36\text{-}4S^2$) and removing two different corners ($36\text{-}X^2\text{-}Y^2$). The results, in the light of the difficulties encountered in high-performing students, can serve as indicators to help educators adapt tasks to mainstream students by analyzing the structures and representations in less-complex situations.

One limitation of this study may be that the data base is very small; the participants all attended the same school, and the inference is that they shared socio-cultural characteristics. As noted in the introductory paragraphs, the authors are keenly aware that the findings are exploratory and cannot be generalized, nor do they have any intention of doing so, in the acknowledgement that the sample is not representative.

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