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Case Study on Intra-Mathematical Connections when Solving Tasks Associated with the Classification of Groups of Prime Order

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Abstract

Isomorphism is a difficult concept to understand for undergraduate students. However, Mathematics Education suggests that it is necessary to promote mathematical connections to foster its understanding. This paper presents some intra-mathematical connections on the classification of groups of prime order that emerged solving task, which were based on a historical and epistemological analysis of the concept of isomorphic groups. This research is a case study. An interview was used for data collection, and qualitative text analysis was performed. Fourteen connections associated with the concepts of group, subgroup, cyclic groups, isomorphism, isomorphic groups, and the Lagrange theorem were identified, involved in the classification of prime order groups. We concluded that the tasks designed with a historical foundation enhance a deep understanding from the connected appreciation of concepts, theorems, methods, and algorithms.

Keywords: Mathematical connections, history, isomorphic groups, mathematics education.

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Estudio de Caso sobre Conexiones Intramatemáticas al Resolver Tareas Asociadas a la Clasificación de los Grupos de Orden Primo

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Resumen

El isomorfismo es un concepto difícil de entender para los estudiantes universitarios. En Educación Matemática, se plantea que es necesario promover conexiones matemáticas para favorecer su comprensión. En este artículo se presentan algunas conexiones intramatemáticas en la clasificación de grupos de orden primo, que emergieron en la resolución de tareas, las cuales, se fundamentaron en un análisis histórico y epistemológico del concepto de grupos isomorfos. La investigación muestra un estudio de caso. Se usó una entrevista para recoger los datos, y para analizarlos se realizó un análisis cualitativo de texto. Se identificaron catorce conexiones asociadas a los conceptos de grupo, subgrupo, grupos cíclicos, isomorfismo, grupos isomorfos y el teorema de Lagrange, implicados en la clasificación de grupos de orden primo. Se concluye que las tareas diseñadas con una fundamentación histórica favorecen en una comprensión profunda a partir de la apreciación conectada de los conceptos, teoremas, métodos y algoritmos.

Palabras clave: Conexiones matemáticas, historia, grupos isomorfos, educación matemática.

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Research on post-calculus mathematics university-level courses has increased, indicating that progress should be made in theoretical understanding of post-calculus learning and teaching (Rasmussen & Wawro, 2017). Especially regarding the concept of isomorphism, in Abstract Algebra, undergraduate students have been reported to have difficulties in proving or disproving that two groups are isomorphic (Lajoie, 2000; Leron et al., 1995; Weber & Alcock, 2004), in recognising the usefulness of this concept (Lajoie, 2000), and in conceptualising isomorphic groups as similar, based solely on properties such as the nature of the elements and operations, the group order and the elements or commutativity (Lajoie, 2000; Leron et al., 1995). Besides, they do not perceive the operation preservation property from this idea of similarity or the formal definition of isomorphism (Lajoie, 2000; Leron et al., 1995; Melhuish, 2018; Weber & Alcock, 2004).

Regarding the classification of finite groups in the first course of Abstract Algebra, students usually must determine how many different groups there are of a given finite order by exploring all possible ways to fill out an operation table and the renaming (Thrash & Walls, 1991). However, this procedure is little convenient for groups of orders greater than four, together with students' difficulties in building operation tables, for example, the tendency to use a canonical procedure and the adoption of a local perspective to reduce the level of abstraction (Hazzan, 2001). In contrast, Cayley's (1854) methods allow a connected appreciation of the underlying concepts, such as group, subgroup, cyclic group, isomorphism, and the Lagrange theorem, which we consider to favour learning since making connections between mathematical ideas is a fundamental part of learning mathematics with understanding (Singletary, 2012).

In this sense, a duality between understanding and connections is visualised because, while students must establish precise conceptual connections to solve tasks with unfamiliar structures and general classes of objects to verify understanding (Melhuish & Fagan, 2018), a solid conceptual understanding is characterised as a knowledge rich in connections (Hiebert & Lefevre, 1986). This highlights that the importance of the study of mathematical connections lies in its link with understanding (Rodríguez-Nieto et al. 2020; Businskas, 2008; Eli et al., 2011; Novo et al., 2019), in part, because connections allow mathematics to be seen as an integrated field. However, in the teaching-learning process, a finished and purely formal presentation of mathematical concepts and ideas is favoured, while epistemological aspects and theoretical construction processes are not prioritised (Arteaga Valdés, 2017), affecting students' understanding of how concepts are interrelated. Therefore, this research aims to characterize intra-mathematical connections that emerge during a college girl student's resolution of tasks associated with the classification of groups of prime order, considering Cayley (1854) as a primary source for their design.

Theoretical Foundation

Mathematical Connections

Mathematics is an integrated field of study, though it is often presented as a collection of separate strands or standards. Nevertheless, as a coherent whole, the students can see the mathematics when they do mathematical connections between ideas and their understanding is deeper and more lasting (National Council of Teachers of Mathematics [NCTM], 2000). In this sense, Armitage (s/f) mentions that connections require students to observe their solutions and reflect, as this procedure is linked to their previous or current learning, generating new learning. This discussion highlights the relevance of mathematical connections, which were defined by the NCTM (2000) as: "the ability to recognize and use connections among mathematical ideas; understand how mathematical ideas interconnect and build on one another to produce a coherent whole; recognize and apply mathematics in contexts outside of mathematics" (p. 64). More recently, García-García and Dolores-Flores (2018) defined mathematical connections as a cognitive process by which a person relates or associates two or more ideas, definitions, concepts, procedures, theorems, representations, and meanings with each other, with other disciplines, or with real life. Likewise, several investigations (Businskas, 2008; Eli et al., 2013; Singletary, 2012) agree that mathematical connections link or bridge mathematical ideas. It is known that this type of connection is divided into extra-mathematics and intra-mathematics. This article will explore only intra-mathematical connections, i.e., those that emerge within the very mathematics and between mathematical entities (García-García & Dolores-Flores, 2018).

Table 1 presents the categorisation used to study mathematical connections, derived from the types of mathematical connections reported in the literature reviewed (Businskas, 2008; Eli et al., 2011; García-García & Dolores-Flores, 2018; Singletary, 2012). In categorical descriptions, the connection components *A*, *B* and *C* correspond to ideas, concepts, definitions, theorems, procedures, representations, or meanings.

Table 1
Categories for different types of mathematical connections

Categories	Description
Different representations	Representations can be *alternative* or *equivalent*. *A* is an alternative representation of *B*, if both are expressed in two different ways (e.g., geometric-algebraic, verbal-algebraic). On the other hand, *A* is a representation equivalent to *B* when both are expressed in two different ways, but within the same form of representation (e.g., geometric-geometric).
Comparison through common features	*A* and *B* share some common characteristics, allowing a comparison based on their similarities or differences (*A* is similar to *B*, *A* is the same as *B*, *A* is not the same as *B*, *A* or *B* defines or describes in a similar way to *C*).
Part-whole relations	When the logical relationships that are established include generalisations and inclusions. The former is of the form *A* is a generalisation of *B*, and *B* is a particular case of *A*. The latter is of the form *A* is included or contained in *B*.
Implication	When a relationship of dependence is established between one concept and the other, where one component of the connection follows logically from another one (If *A*, then *B*, If *A*, then *B* and not *C*).
Procedure	A mathematical or algorithmic procedure is associated with a particular concept (*A* is a procedure used to work with *B*).
Characteristic/Property	It is established when defining some features or describing the properties of concepts in terms of other concepts that make them different from or similar to the others.
Derivation	It manifests itself when knowledge of a concept is used to construct or explain another concept; although it is not limited to the recognition of some derivation.
Connecting methods	It refers to the consideration of multiple methods of solving a problem, i.e., *A* or *B* can be used to find *C*.
Reversibility	It is the ability to recognise and establish bi-directional relationships between mathematical ideas. For example, when starting from a concept *A* to arrive at a concept *B* and reversing the process starting from *B* to return to concept *A*.
Meaning	It refers to the sense that an individual gives to a mathematical object, so the attributed meanings may be limited by its definition or the context of its use.

History As a Tool

In Mathematics Education, we identify several arguments for the use of history and how it can be used in the processes of its teaching and learning (Furinghetti, 2020; Jankvist, 2009). Based on Jankvist's (2009) categorisation of why history may or should be used in mathematics teaching, this research considers the use of history as a *tool*, since it plays an important role in supporting the teaching and learning of mathematical concepts, theories, methods, and algorithms (*in-issues*). Specifically, the study of the sources in mathematics learning, not addressing the study of the history of mathematics directly but indirectly, i.e., without explicitly discussing historical development (Jankvist, 2009), served as a foundation for task design and its role in establishing mathematical connections was recognised.

The classification of groups of prime order

In *On the theory of groups*, *as depending on the symbolic equation* $\theta^n = 1$, of 1854, Cayley proposed the classification of finite groups according to their form and based on an approach of generators and relations for groups, exemplified the distinction between the ordinary equation $x^n - 1 = 0$ and the symbolic equation $\theta^n = 1$. He also considered a finite group $1, \alpha, \beta, \gamma, \ldots$ (n different symbols, where 1 is identity) as a system of roots of the symbolic equation $\theta^n = 1$, explored the nature of n in that equation, and concluded that every finite group G of prime order (*index*) is cyclic (without using this term): "when n is a prime number, the group is of necessity of the form $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$, ($\alpha^n = 1$)" (p. 41). Likewise, he stated that if G is a cyclic group of prime order (in the current terminology), then every element of G, except for the identity, is a generator (*prime roots*); while for a cyclic group of composite order n, there will be as many generators as natural numbers $k < n$, where k and n are relatively primes.

Cayley stated that any group of order n, where n is a prime or composite number, of the $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$, ($\alpha^n = 1$) (cyclic) form, "is in every respect analogous to the system of the roots of the ordinary binomial equation $x^n - 1 = 0$" (pp. 41-42), i.e., isomorphic to the system of n^{th} root of unity. Thus, any cyclic group of order n will behave in the same form as the latter. In particular, all groups of prime order *have the same form* as the cyclic group of

n^{th} roots of unity, and for any given prime p, there is *essentially* only one group of order p.

Methodology

This research has a qualitative approach (Creswell, 2014) because it explores the mathematical connections' cognitive attributes of an individual when solving tasks associated with the classification of groups of prime order. The research design was a case study whose characteristic is to analyse the context and processes that clarify the theoretical issues being studied (Njie & Asimiran, 2014). The data was collected through a questionnaire and an interview.

To select the case study, we followed up a group of undergraduate students of the fifth semester (20 to 21 years-old) of Mathematics, who had started a first course of Modern Algebra (Abstract Algebra). With the course teacher's consent, we watched the classes, took notes of the contents developed, the examples given by the teacher, the type of tasks and students' participation.

The case was chosen according to two criteria: i) the student should complete the Abstract Algebra course, and ii) wished to collaborate in the investigation voluntarily. This choice led us to select our study case, student Lu, who had not succeeded in the Abstract Algebra course but had participated enthusiastically, it is worth mentioning that she had completed two Calculus courses, one on Mathematical Analysis, and two on Linear Algebra.

The Questionnaire

We developed a questionnaire that incorporated a sequence of eight tasks of an intra-mathematical nature and was validated both by an expert in the area of Abstract Algebra with more than ten years of teaching experience in a Bachelor's Degree in Mathematics and by users, considering the results of a pilot test applied to five students of the fifth semester of which Lu was a participant.

The tasks of the questionnaire are described below (see Figure 1):

1. Given the following groups:
a) $(S_2, \circ)$;
b) $(\mathbb{Z}_3, +_3)$;
c) $G = \{(1),(1\ 2)(3\ 4),(1\ 3)(2\ 4),(1\ 4)(2\ 3)\} \subset S_4$, with the composition;
d) $(\mathbb{Z}_5, +_5)$;
e) $(S_3, \circ)$;
f) $(\mathbb{Z}_6, +_6)$;
g) $(\mathbb{Z}_7, +_7)$.
i. How many subgroups do each of them have? Explain your answer.
ii. Calculate the order of each subgroup.
iii. What is the relationship between the group order and the order of any of its subgroups? Explain your answer.
2. How do you determine all the subgroups that a prime order group has? Explain your answer.
3. If G is a prime order group, p and x is an element of G, what can be said about the subgroup generated by x?
4. Consider $G = \{1, i, -1, -i\} \subset \mathbb{C}$, with the multiplication of complex numbers and $x \in G$.
i. What can you say about the generated by x?
ii. What is the relationship between the order of x and the order of the generated by x?
5. Consider $G = (\mathbb{Z}_5, +_5)$ and $x \in G$.
i. What can you say about the generated by x?
ii. What is the relationship between the order of x and the order of the generated by x?
6. Consider $G = (\mathbb{Z}_8, +_8)$ and $x \in G$.
i. What can you say about the generated by x?
ii. What is the relationship between the order of x and the order of the generated by x?
7. *i*. Consider a group G of order three with generating element α. Build the operation table of G.
ii. Consider a group H of order three with generating element β. Build the operation table of H.
iii. Compare both tables and determine if groups G and H behave in the same way.
iv. Could you give an example of a group of order three that does not behave in the same way as the previous ones? Explain your answer.
8. *i*. Consider a group A of order five, with generating element γ. Build the operation table of A.
ii. Could you give an example of a group of order five that does not behave in the same way as the previous one? Explain your answer.

Figure 1. The interview questionnaire.

Task 1 aims to explore a student's knowledge of the Lagrange theorem. Examples of finite groups employed included integers module n [$(\mathbb{Z}_n, +_n)$] and permutations of a set X [symmetrical group of n letters, $(S_n, \circ)$]. For the presentation of the groups, the order from lowest to highest from two to seven is taken into account, and cyclic and non-cyclic groups are considered.

Tasks 2 and 3 are intended to explore a student's reasoning regarding the subgroups of a group of prime order. In Task 3, the term *generated* is used without explicit reference to a *cyclic group*; specifically, it is expected that the student can establish relationships between a group of prime order and the set generated by any of its non-identity elements.

Tasks 4 to 6 consider finite cyclic groups, and the student is expected to associate that the term generated by an element x of the group [$\langle x \rangle$] is a subgroup and that the order of an element of the group is equal to the order of the subgroup generated by this element. In relation to tasks 1 to 6, we should mention that they involve understanding preliminary concepts to face the specific tasks on the classification of groups of prime order.

Tasks 7 and 8 aim to explore the relationships between ideas and mathematical results that a student establishes in determining that the only group of (prime) order p is the cyclic group of that order. The student was expected to relate the knowledge that emerges in the previous tasks (1 to 6), which, comprehensively, aims to explore the associations between ideas and mathematical results that the student can establish related to the concepts involved and required in the classification of groups of prime order.

The Interview

The interview was used as an information-gathering tool to deepen the reasoning of the case. According to Arnon et al. (2014), based on the student's answers, the interviewer can choose a more didactic route, since the aim of an interview is to determine and explain how individuals build their understanding of mathematical concepts and allows the interviewer to observe the construction process as it develops. Also, if the student gets stuck in a specific task or does not provide a reasonable answer to a question, clues can encourage their progress in constructing concepts and motivating connections between different notions (Oktaç, 2019).

In the case study, the interview questionnaire was applied in three 90-minute sessions each. The interviews were recorded in audio and video for further analysis. In addition, they were transcribed in their entirety to be analysed together with the written productions.

Data Analysis

For data analysis, we used the qualitative text analysis method (Kuckartz, 2014), which consists of the following phases: *reading and interpreting the text*, *building categories*, *coding segments of the text*, *analysis*, and *presentation of results*. In the phases, the categorisation of mathematical connections is considered.

Phase 1. *Reading and interpretation the text.* Familiarisation with the data was established from the reading and analysis of the transcripts, along with the written productions based on the research aim.

Phase 2. *Building categories*. Based on the research aim, the construction of categories was carried out deductively, i.e., before collecting the data and based on the established framework on mathematical connections. In this sense, the following main categories (typology of mathematical connections) were considered: *different representations*, *comparison through common features*, *part-whole relations*, *implication*, *procedure*, *characteristic/property*, *derivation*, *connecting methods*, *reversibility*, and *meaning*.

Phase 3. *Coding segments of the text.* The encoding of the data was carried out from the main categories, i.e., the categories established in the second phase were given codes. Specifically, a search for words or phrases in the transcripts associated with the typology of mathematical connections was performed, and the codes were accordingly assigned.

Phase 4. *Analysis.* Based on the third phase results and triangulation between the three authors, the specific mathematical connections were characterised concerning the categories for which evidence was found, i.e., based on a discussion and consensus of their correspondence with the data.

Phase 5. *Presentation of results.* For their presentation, the mathematical connections were grouped according to the mathematical concept associated with them: group, subgroup, Lagrange theorem, cyclic groups, isomorphism, and isomorphic groups. These connections fit into one or more of the categories proposed by Businskas (2008), Eli et al. (2011), García-García and Dolores-Flores (2018), and Singletary (2012).

Results

This section presents the characterisation of each of the fourteen intra-mathematical connections identified from Lu's productions when solving the tasks posed, and which are later denoted by C_i, i = 1, 2, …, 14 . Such connections are associated with the mathematical concepts: group, subgroup, Lagrange theorem, cyclic groups, isomorphism, and isomorphic groups. Only nine of the ten types of connections were identified, which fell into one or more categories of the following: *different representations*, *comparison through common features*, *characteristic/property*, *connecting methods*, *procedure*, *part-whole relations*, *implication*, *derivation*, and *meaning* (see Table 2).

Table 2
Mathematical connections identified in tasks solved by Lu

Concepts	Mathematical connections	Type of connection
Group and subgroup	(C_1) A subgroup is a group whose elements are contained in the main group.	Meaning Derivation
	(C_2) A subset is a subgroup with the group's restricted operation.	Characteristic/Property
	(C_3) To find the subgroups of a finite group, the subsets that satisfy the group properties are verified one by one.	Procedure Different representations Connecting methods
	(C_4) Each group has as subgroups the trivial and the total.	Implication
	(C_5) The identity is an element that belongs to any subgroup.	Characteristic/Property
Lagrange theorem	(C_6) Any finite subgroup order divides the group order.	Implication
	(C_7) A group *G* of prime order has as subgroups only the trivial and the total.	Implication Derivation
	(C_8) The order of any element divides the group order.	Comparison through common features

Table 2 (continue)
Mathematical connections identified in tasks solved by Lu

Concepts	Mathematical connections	Type of connection
Cyclic group	(C_9) The generator of a finite group G is an element that, when applied n times, all the elements of G are obtained.	Meaning
	(C_{10}) A group of prime order is necessarily cyclic.	Implication
Isomorphism and isomorphic groups	(C_{11}) The table of a group G with generating element α is similar to that of the group ($\mathbb{Z}_3$, $+_3$).	Comparison through common features Derivation
	(C_{12}) Groups of order three are similar.	Comparison through common features Derivation
	(C_{13}) Two groups of order five can be seen as the same group changing names to preserve operations.	Derivation
	(C_{14}) There is only one group of prime order, except for the denomination of the elements.	Part-whole relations

Below is the analysis of some extracts of the three interview sessions where the fourteen identified connections are shown, as well as the type of connection, regarding mathematical concepts with which they are associated: group and subgroup, Lagrange theorem, cyclic groups, isomorphism, and isomorphic groups.

Mathematical Connections Associated with the Concepts of Group and Subgroup

Understanding the concept of group isomorphism involves, among other things, understanding the concept of group, while the concept of subgroup is derived from it and can be intuitively considered as one group that is contained in another, taking into account that the operation must coincide. In particular, when Lu was asked to determine the subgroups of specific finite groups (task 1), we identified five mathematical connections, which we explain below:

(C_1) A subgroup is a group whose elements are contained in the main group. Lu associated the term subgroup with the meaning of inclusion. To explain what a subgroup is, she based her knowledge on the concept of group, specifically, on the fulfilment of axioms: the existence of an identity element, inverses, closure, and associativity. So, Lu also established a derivation-type connection. The following episode alludes to the discussion about subgroups of $(S_2, \circ)$.

> Interviewer: You are asked to determine how many subgroups $(S_2, \circ)$ has. What is a subgroup?
>
> Lu: It is a group whose elements are contained in the largest group. Well, would it be [$(S_2, \circ)$ it is a subgroup] ... $\mathbb{R}$ is it a subgroup of $\mathbb{R}$? [refers to group $(\mathbb{R}, +)$] … $\mathbb{R}$ would it be a subgroup of $\mathbb{R}$? If so, we have a subgroup that is the subgroup itself ... if there were, more subgroups they would be each of the elements, but that means that each element must comply with all the group properties ... then only the identity [refers to {(1)} as another subgroup of S_2].

(C_2) A subset is a subgroup with the group's restricted operation. Lu became aware of this connection after working with the different groups included in task 1, especially working with whole modules *n*. Said connection emerged from connection C_1, when the student associated the concept of subgroup with the fulfilment of the closure property and deduced that a subset should be a subgroup with the restricted operation of the group. In this sense, Lu recognised a *feature* that constitutes or is part of a subgroup.

> Interviewer: You say that one subgroup of $(\mathbb{Z}_3, +_3)$ is the subset that contains the identity and another is $\mathbb{Z}_3$. ... Are they all subgroups of $(\mathbb{Z}_3, +_3)$?
>
> Lu: Let's see, it must be a subset, and it must be a group with the operation, here [the subset {0, 1}] would be with sum 2 [it means adding module 2] or with sum 3? In my opinion, it could be subgroup if it is with sum 2, if it is with sum 3 it is not subgroup, but then the other option is {1, 2} or {0, 2} [subsets of $\mathbb{Z}_3$], they are not subgroups with sum 3, too, because here it would be 2 and here it would be 0 and 0 does not belong here. And here, we would have {0, 2} and 1 doesn't belong here.

(C_3) To find the subgroups of a finite group, the subsets that satisfy the group properties are verified one by one. For groups of order *n* that Lu investigated, she verified for all subsets of possible combinations with one, two, up to *n* elements of the group in search of those that fulfilled the axioms of a group. This *procedure*-type connection was identified when Lu determined all subgroups of a given group and emerged when C_1 and C_2 were settled. She also used *alternative representations* to refer to a subgroup. Besides the informal definition of this concept, Lu relied on the operation tables to decide whether a subset was a subgroup (*connecting methods*). We show the following interview episode, for the specific case of determining all subgroups of $G = \{(1),(1\ 2)(3\ 4),(1\ 3)(2\ 4),(1\ 4)(2\ 3)\} \subset S_4$, with the composition.

> Lu: How many subgroups does G have? Let's see, is this [{(1 2) (3 4)}] a subgroup? Yeah, right?
> Interviewer: How do you know?
> Lu: (1 2) (3 4) composition (1 2) (3 4) ... the 4 sends it to the 3, and the 3 sends it to the 4, it is the identity in S_4.
> Interviewer: Is [{(1 2) (3 4)}] a subgroup?
> Lu: No, because it's not there. Then the same thing would happen here [(1 3) (2 4)], because if you send it and I apply it again and that same one returns it. And with this [(1 4) (2 3)], it would be the same. Of two [elements], if I apply identity [(1)] and * [* = (1 2) (3 4) in Figure 2], here [when operating * and *] we already saw that it gives us the identity and here [when operating (1) and *], it would give us * and here the identity [when operating (1) and (1)]. But if this [(1 3) (2 4)] we, do it **, the 1 sends it to the 3 and the 3 returns it [{**, (1)} , it is another subgroup of two elements. Also {(1), (1 4) (2 3)}]. And of three [elements]…

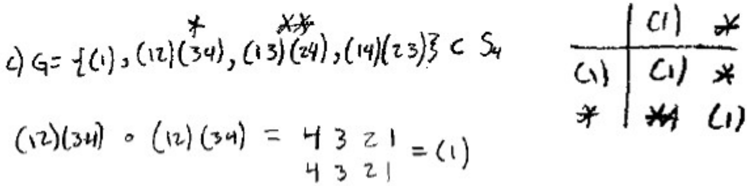

Figure 2. Use of the operation table to verify whether {(1),(1 2) (3 4)} is a subgroup.

(C$_4$) Each group has as subgroups the trivial and the total. This connection was established when from the connections C_1, C_2, and C_3, Lu deduced that the subset containing the identity element and the group itself were subgroups, which she called "neutral" and "total", during the determination of the subgroups of the groups proposed. Lu made an implication-type connection by recognising trivial and total subgroups as subgroups of any group.

> Lu: For any group, the neutral is a subgroup, and a group can be a subgroup of itself.
> Interviewer: How could you be sure?
> Lu: Well, if it is a group, it will fulfil them [the axioms of a group].

(C_5) The identity is an element that belongs to any subgroup. This connection was established when Lu highlighted the singularity of the identity element in a group and concluded that combinations of elements (the subsets) that did not contain it, could not be considered as possible subgroups from the fulfilment of the axioms of a group. Therefore, Lu recognised a *feature* that constitutes or is part of a subgroup.

> Interviewer: What is the relationship between the order of the group and the order of any of its subgroups?
> Lu: Well, at most, they can be the group order because the group is a subgroup of the group ... and a subgroup of cardinality one, too. Because well, I just sensed that the operation we are going to occupy is that of the main group, if that is so, any other element of the group can't act like the identity, and in the subgroup, we need an identity. ... For it to be a group, a subset must have an identity, so how is it going to be a subgroup if we remove the identity? Another element would need to act as identity, but if another element acts as identity, then there would be two identities.

Mathematical Connections Associated with the Lagrange Theorem

The relationship between a group order and the order of its subgroups is expressed by the Lagrange theorem. From this theorem, it is also possible to deduce the relationship between a group order and the order of its elements. The three mathematical connections that Lu established with this theorem are presented below. These connections emerged from her failure to initially consider establishing an *equality* relationship when operating subgroup orders to, somehow, obtain the group order.

(C_6) Any finite subgroup order divides the group order. During the discussion of the relationship between a group order and its subgroups of task 1, Lu established this *implication*-type connection from observing and analysing the table she built with the seven groups and their respective subgroups where in *all* cases, the order of any subgroup *divided* the group order. During the interview, Lu was also questioned about the possible subgroups of a group of order eight, where she showed that she was aware that a group could not have a subgroup whose order does not divide the group order.

> Interviewer: What is the relationship between the group order and the order of any of its subgroups?
> Lu: [Builds a table with the groups and their respective previously determined subgroups, see Figure 3] All you ask is that by dividing them by an integer?
> Interviewer: What does that mean?
> Lu: It would be the group order between the subgroup order, integer, ... maximum group order. ... They are, what are the names of the numbers that divide another number? Divisors? Yes.
> Interviewer: For example, for a group of order eight, what order could the possible subgroups be? ... Is it possible there's one of order four?
> Lu: Yes, I think so and of order two maybe also, although we cannot be sure, it depends on the group.
> Interviewer: Why do you say that a group of order eight can have a subgroup of order four or two?
> Lu: Because the four divides eight, and the two divides eight. I mean, what I'm saying here is that there's not going to be one in order five in order eight because five doesn't divide eight.

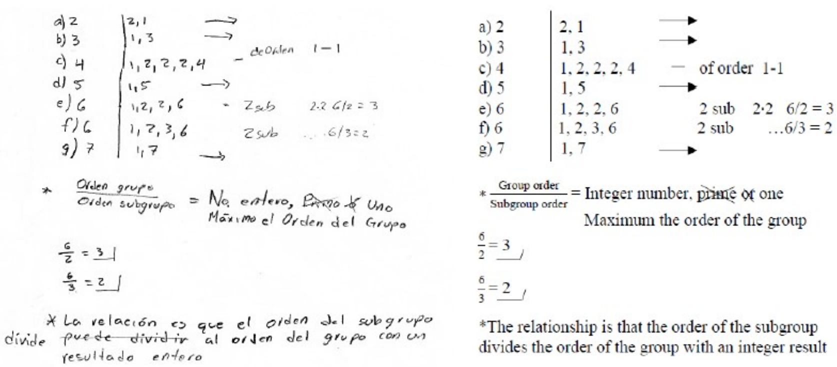

Figure 3. Comparative table between the group order and the order of its respective subgroups.

(C_7) A group G of prime order has as subgroups only the trivial and the total. This connection emerged from the relationship that she managed to establish between the orders of a group and any of its subgroups (see connection C_6). Lu argued that, for groups of prime order, the only subgroups they have are the trivial and the total. Therefore, she made *derivation* and *implication*-type connections.

> Interviewer: How do you determine all the subgroups that a group of prime order has?
> Lu: If what I'm saying is that the order of the subgroup divides the order of the group, we need a number that divides, that can divide that order of the group, but if the group order is prime, they would be 1 and the same number. So, ... for 1 to divide it, it means that it is of order one, that it is the neutral, and that the same number divides it, it means it is the total.

(C_8) The order of any element divides the group order. This connection emerged after the discussion about the generators of a group of prime order and the establishment of the connection C_{10}, when Lu became aware of the relationship between the order of the generating element and the order of the subgroup that it generates from the resolution of tasks 4, 5, and 6. Lu was based on the feature of equality that she observed between the respective orders, concluding that the order of an element divides the order of the group, making a *comparison*-type connection *through common features*.

> Interviewer: In the next task, given $G = (\mathbb{Z}_5, +_5)$ and $x \in G$. What can you say about the generated by x? What is the relationship between the order of x and the order of the generated by x?
> Lu: Well, this [refers to $(\mathbb{Z}_5, +_5)$] is of prime order, so it's only going to have [subgroups] the neutral and the total. ... The order of the [generator] element is equal to the order of the subgroup it generates. I had said before that the order of the subgroup can divide the order of the group; if it is the same as that of the order of the element, then the order of the element can divide the order of the group.

Mathematical Connections Associated with the Concept of Cyclic Group

A group is cyclic if it can be generated by one of its elements (generator). In the finite context, the elements of the group are obtained by operating the generator several times with itself. In particular, in a group of prime order, every element, except identity, is a generator. The two mathematical connections that Lu established with this concept were the following:

(C_9) The generator of a finite group G is an element that, when applied n times, all the elements of G are obtained. This connection arose from reflecting on task 3 and was subsequently applied in tasks 4, 5 and 6 (see Figure 4) involving finite cyclic groups. The *meaning* that Lu attributed to the *generating* term was that of an element from which all the elements of the group are constructed. Specifically, from successive powers of an element, which at some point produces the identity. These powers can be positive, negative or zero.

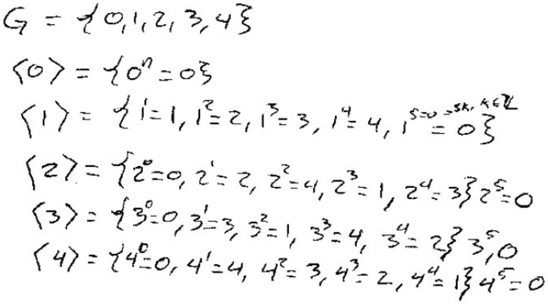

Figure 4. The generating elements of the group ($\mathbb{Z}_5$,$+_5$).

Interviewer: What does it mean that an element is a generator of the group?

Lu: The generator is that you take an element, and you will apply it n times, and you get all the elements of the group (see Figure 5).

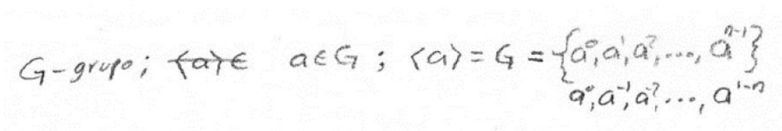

Figure 5. The generator is an element from which all the elements of the group are constructed.

(C_{10}) A group of prime order is necessarily cyclic. This connection emerged when Lu established the connection C_7 with the feature of the subgroups of a group of prime order. Faced with the question, what can be said about the subgroup generated by an element *x* of *G*, if *G* is of prime order? Lu noted that the one generated by the element *x* was equal to the total subgroup, due to the impossibility that the identity element generates *G*. Therefore, the only restriction was that element *x* was different from the identity. Lu stated that all elements, except for the identity, were generators of the group, establishing an *implication*-type connection.

> Interviewer: If G is a prime order group, p and x is an element of G, what can be said about the subgroup generated by x?
> Lu: The set generated by x is subgroup, then [she writes $< x > =$ G]. For example, in ℤp, what elements can ℤp generate? Zero cannot generate it because, when applied several times, zero will always give zero. I don't know which element is x, but we have exceptions for it to be subgroup or subgroups of G. We have two, one is this [trivial], and the other is this [total]. If it is a subgroup generated by x, it means that x is an element of the group. In this case, it would be an element ... that generates a subgroup, and since we only have two possibilities of subgroup, then the one generated by x is G.

Mathematical Connections Associated with the Concepts of Isomorphism and Isomorphic Groups

Understanding the concept of isomorphism is central to classify groups of prime order (*p*): there is only one group of a given prime order (the cyclic order group *p*). In other words, all groups of order *p*, where *p* is a prime number, are isomorphic. Here are three mathematical connections associated with the concepts of isomorphism and isomorphic groups:

(C_{11}) The table of a group G with generating element α is similar to that of the group ($\mathbb{Z}_3$, $+_3$). Task 7 required constructing the operation table of a group *G* of order three with generator element *α*. Lu was expected to associate the idea that, if *α* is a generator of the group, from *α*, all the elements of the group, i.e., the total subgroup, can be obtained. This task was not easy for Lu, who established *derivation* and *comparison connections through common features* when she relied on knowledge of the properties of the group ($\mathbb{Z}_3$,$+_3$) and its table to construct that of group *G* and, from this, concluded that they were similar (see Figure 6).

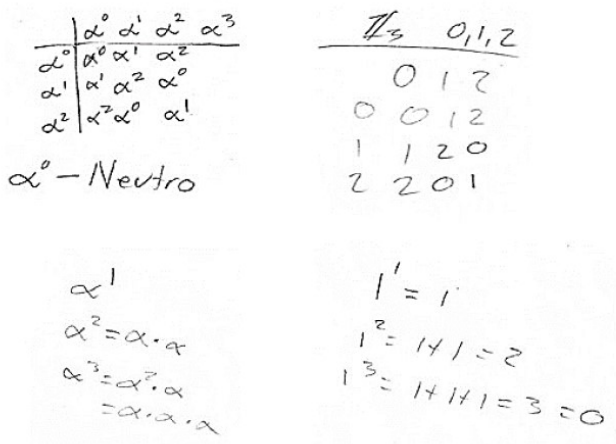

Figure 6. Table of a group G of order three with generating element α based on group ($\mathbb{Z}_3$,$+_3$).

Interviewer: Consider a group G with generating element α. You are asked to build the table of said group.
Lu: One could be $\mathbb{Z}_3$ with 0, 1, and 2 [constructs table of ($\mathbb{Z}_3$,$+_3$)]. ... The elements would be α^0, α^1 and α^2. Because whatever is generated, I understand that we have α and then we start α^0 that it would be like 1[in ($\mathbb{Z}_3$,$+_3$).... α^0 is the identity. ... Concerning $\mathbb{Z}_3$, the identity, and I have two elements, 1 and 2.

(C_{12}) Groups of order three are similar. In the construction of another group *H* of order three with generator element *β*, Lu claimed that the tables of *G* and *H* were the same/similar. She established a *comparison*-type connection *through common features*, since her main argument was based on the appreciation of similarities between the groups, such as the order of the elements and an arrangement of the position of the elements in the operation tables.

Interviewer: You constructed two tables that correspond to groups of order three, G and H, with generator elements α and β. Compare both tables and determine if groups G and H behave in the same way.
Lu: I say they are similar. We have an identity in both cases ... and in both cases, it is at power three that this identity comes out and we have two other elements that are inverse of each other.
Interviewer: How can you be sure they are similar?

Lu: Through the table. Because if we do here $\alpha^3 = \alpha^0$, then either α^3 or α^0 is the identity, and if we change the symbol, it would make the table more similar. ... And so, it becomes more evident that the tables are the same (see Figure 7).
Interviewer: Even if they have α and β elements?
Lu: Yes. As we told you [the term used by the teacher] when the tables were similar, that we could change their names? Isomorphic? Something like that.

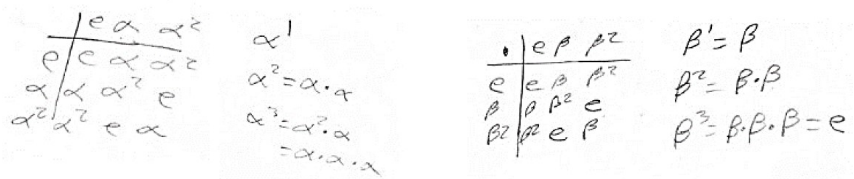

Figure 7. Two similar groups of order three.

Lu also made a *derivation*-type connection when she used her knowledge on one group to determine that any other of order three should have a table similar to *G* and *H*. Her reasoning shows that she considered $\mathbb{Z}_3$ groups *G* and *H* as *really the same group with different names for the elements and operations*, from the appreciation of the structural property of the order of the elements.

Lu: Will there be another group of order three other than $\mathbb{Z}_3$? No, because if there were a table where we had identity, identity, identity [see the diagonal of the table in Figure 8]; element one and element two [fill the table]. ... Is this a group?
Interviewer: Would that be another group of order three?
Lu: [verifies the axioms of a group from the table, see Figure 8] and it would have to be associative. ... No, it is not a group.
Interviewer: And if you consider a group of order three with generator element γ.
Lu: Well, the table would be similar to the two previous tables [G and H]. The elements that are obtained from γ are γ^1, γ^2 and γ^3, which would be γ, γ^2, and the identity.

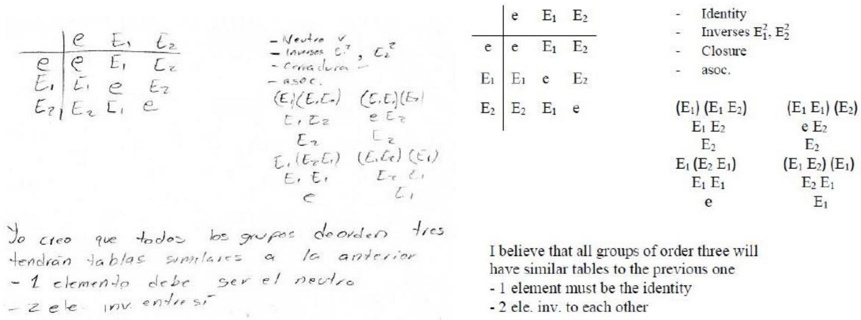

Figure 8. All groups of order three have tables similar to G and H.

(C_{13}) Two groups of order five can be seen as the same group changing names to preserve operations. This connection was identified during the construction of the operation table for a group of order five *A* with generator element γ (task 8), which requested an example of another group of order five that was not similar to group *A*.

Lu focused on showing that there is only one group of order five, and any other group would be similar. Besides considering the ownership of the order of the elements, she associated the idea of similar groups with the search for *a relationship* that by *changing the name* of the elements, she could show that there is only one group of order five. Not only was she able to give such a relationship; she also knew why that renaming proposed worked, establishing *derivation*-type connections, since her argument was based on both the property of isomorphism, which allows the renaming of the elements through the biunivocal correspondence so that the operation tables look "similar", and the fact that the tables were "similar" ensures that the preservation of the operations is satisfied.

> Interviewer: Could you give an example of a group of order five that does not behave in the same way as the previous one [group A]? ... How did you establish that relationship [between groups A and G]?
> Lu: γ^2 with b because it is a^2, c is a^3 and d is a^4.
> Interviewer: Could you explain how you are operating between the elements?

Lu: We have γ^2 to correspond to b and we have γ ... and γ is a (see Figure 9). So, we have a operation b, which equals a^2 with a, which is equal to a^3, which is c, and c by the relation is this [γ^3].
Interviewer: So how do you know that the change you made actually shows that the tables are the same with elements denoted with different symbols?
Lu: Because it is the same, exactly the same ... it is fulfilled for everyone. It is fulfilled because the relationship is fine.
Interviewer: Is this other group G different from group A?
Lu: No. That is, in strict order, they are not the same because they are not the same elements, but they behave similarly. So, if we look at it for behaviour, they are the same.

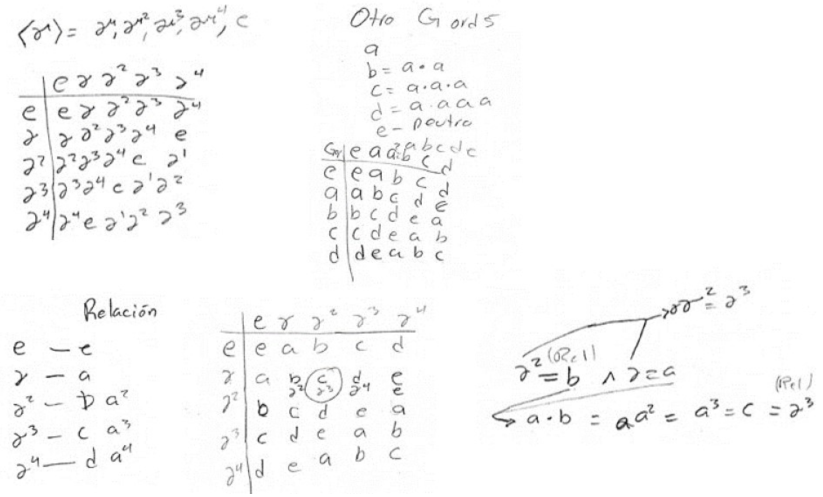

Figure 9. Groups of order five are similar.

(C$_{14}$) There is only one group of prime order, except for the denomination of the elements. This connection emerged from the connections C_{12} and C_{13}, when the student generalised it for any group of prime order. Lu associated the result that, in a group of prime order, all elements different from the identity are generators, and from that element, it was possible to build its table of operations. She showed that the tables were "similar" and, therefore, the groups were "similar", establishing a *part-whole relations*-type connection, since she recognised groups of order three and five as specific instances of a finite group of prime order. Lu concluded that, in general, there is only one group of prime order given, except for the denomination of the elements.

> Interviewer: How many groups are there in a given prime order p?
> Lu: That its order is prime means that all the elements it has, except for the identity, will generate it, which means that one can take any of its elements and build its table ... the table of an arbitrarily selected element... That is, we have all [groups of order p denoted differently], I arbitrarily take one of each of them [elements], I build their table, and I see that the tables behave similarly, i.e., it is the same ... there will always be one [a single group of a given prime order].

Conclusions

This research reports the intra-mathematical connections included in the classification of groups of prime order from a case study. Five connections associated with the concepts of group and subgroup were identified, three related to the Lagrange theorem, two with the cyclic groups, and four with the concepts of isomorphism and isomorphic groups. Each of the fourteen mathematical connections was associated with one or more categories (types of connections).

Results indicate that, on most tasks, Lu became aware of the mathematical connections established once she solved them or reflected on the results and procedures performed. Although Lu made connections between concepts, definitions, and procedures when approaching the tasks, her knowledge of the concepts and the mathematical results underlying the classification of groups of prime order was limited, which could explain the insubstantial use of connections of the types: *different representations*, *characteristic/property*, and *connecting methods*. In this sense, the results show that in the task solving process, Lu was discovering, building, and using her new knowledge to move forward, which explains the frequency of derivation and implication-type connections.

The type of tasks that considered specific groups favoured exploration and discovery from the establishment of *comparison*-type connections *through common features, characteristic/property*, *derivation*, and *implication*, which agrees with the ideas of Hazzan and Zazkis (1999), who point out that the construction of significant mathematical notions requires recognising the similarities of general ideas in different particular examples to discover their structure and common attributes.

On the other hand, Lu established a *derivation*-type connection (C_{13}) when she recognised that two groups of order five could be seen as the same group by changing names to preserve the operations. In other words, she used the concept of isomorphism as bijective homomorphism and associated the idea of similar groups with the search for a relationship that, by renaming the elements, could show that there is only one group of order five. This result differs from other studies that show that the students' conceptualisation of isomorphic groups as similar groups is associated with a literal interpretation of this word, while preservation of operations is not perceived from the idea of similar groups (Lajoie, 2000; Leron et al., 1995). However, we could not be sure whether Lu saw isomorphism as an equivalence relationship, which allows classifying the groups "*up to isomorphism*."

It also shows a dependence between connections, in the sense that, for example, to establish the connection C_{14} , it was essential for Lu to make the connections C_{12} and C_{13}. In other words, the connection "there is only one group of prime order, except for the denomination of the elements" was derived from Lu's generalisation of the specific cases of groups of order three and five: "groups of order three are similar" and "two groups of order five can be seen as the same group changing names to preserve operations", respectively. This dependency is related to the student's level of understanding and the use of that knowledge in solving the tasks posed.
Finally, from this research, an educational implication from the identification of mathematical connections corresponds to the design of tasks to establish explicit connections to strengthen the understanding of the concepts and results underlying the classification of groups of prime order. It also highlights the use of mathematics history and epistemology to favour the presentation of concepts, theorems, methods, and algorithms (*in-issues*) connected to students about the problems and ideas that generated them.

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