ESTUDIO 2: FOURTH-GRADERS' JUSTIFICATIONS IN EARLY ALGEBRA TASKS INVOLVING A FUNCTIONAL RELATIONSHIP[15]

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Abstract

In the context of early algebra research and as part of a classroom teaching experiment (CTE), we investigated fourth grade (9- to 10-year-old) students' justifications of how they performed tasks involving the functional relationship y = 2x. We related their written justifications (part of the task) to the task characteristics, which included various semiotic systems (verbal, numerical and alphanumeric, among others) and the demand of different type of justifications. The role of classroom discussion in helping express the functional relationship orally in more sophisticated terms was also investigated. The findings showed that students' written justifications changed with the semiotic system involved in the task. Oral discussion helped students generalize in more sophisticated terms than in their written justifications, in which they omitted information or used less precise language.

Keywords: Early algebra, generalization, justification, functional thinking, semiotic system

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1. Introduction

One of goals of early algebra research is to *algebrify* elementary mathematics considering several dimensions of algebra: (a) generalized arithmetic; (b) the study of patterns; (c) equivalence, expressions, equations and inequations; and (d) the study of functions (Molina & Mason, 2009; Blanton et al., 2011). In addition, it includes processes that are transversal to those contents: generalization, representation, justification and reasoning about generalizations.

The object of this study is the process of justification in the context of early algebra tasks involving functional relationships. We focused on functional thinking, understood to be generalizing, representing and justifying relationships between covarying quantities, as well as the use of representation to predict and understand how variables behave (Blanton & Kaput, 2011). This type of thinking allows integrating algebra in the elementary mathematics curricula without adding any new content, just interpreting arithmetic operations as functions. It is also a useful resource in problem solving; it allows to deal with functions, in students' daily contexts, as a variation (e.g., Carraher & Schliemann, 2007, 2015).

Justification, a skill developed gradually (Stephens et al., 2017), is a way of determining and explaining the truth of a conjecture or assertion. We are interested in its study because encouraging students to justify their thinking helps them understand and actively participate in the construction of mathematical concepts and processes (Chua, 2016).

Justification is an act of communication. Students' mathematical knowledge can be analyzed from what they say or their use of other signs (Morgan et al., 2014). Furthering mathematical communication helps students express themselves more clearly. It also helps teachers understand what students are thinking and make better informed pedagogical decisions (Ingram et al.,2019). As acknowledged in curricular guidelines, Spanish elementary school students are expected to "verbally express and reason the process followed in solving a problem […]; formulate conjectures and find arguments to validate or refute them" (Ministerio de Educación, Cultura y Deporte, 2014, p.19388).

In the context of a CTE designed to explore and foster functional thinking in elementary school students, we sought to determine how justification may be conditioned by the characteristics of the tasks proposed and describe the development of students' ability to justify. We delimited the research problem in terms of the following questions:

> What are the characteristic features of written justifications in functional tasks involving different semiotic systems and demanding various types of justifications in generalization?
> How might oral justification arising in group discussion promote further more sophisticated expression of inter-variable relationships?

Considering that the type of arguments wielded depends on students' skills and the nature of the task (Chua, 2016), we include in the tasks the demand of two type of justifications: elaboration and validation. At the same time, we consider that semiotic systems include conventional representations systems but also non-conventional systems such as gestures, rhythm and natural language. By using them students give meanings to mathematical objects (Radford, 2002). A higher sophistication in the expression is related to the idea of semiotic contraction, i.e., the reorganization of semiotic resources produced as a result of students' higher awareness of mathematics meanings and interpretations (Radford & Sabena, 2015). The semiotic contraction consists of making a choice between what counts as relevant and irrelevant (Radford, 2018)

The interest of studying justification and the expression of generalization considering different semiotic systems is manifested by Kaput (2009) as an open line of research: "Given the essential role of argument and expression in generalization, and the fact that younger learners need to use natural language and other naturally occurring forms of expression, my sense is that we have much to learn about generalization and hence the development of algebraic thinking, from studies of gesture and talk—including intonation (p.213)". In that paper the author claims that theoretical constructions such as that of Radford (2009) are needed to get a deeper understanding of how speech, gesture, and the many different systems of signs interact, particularly if we adopt his perspective that knowledge objectification is almost always, particularly in education, a multimodal, semiotically mediated phenomenon. Considering this concern, in this study we aim to contribute to widening the study of functional thinking by adopting a multimodal view of thinking. We consider thinking not just as a mental activity but as a process mediated and evidenced by language, gestures, rhythm and all the resources used to interact with the environment. We understand that "the source of abstract mathematical thinking is to be found in the sophisticated linkage of language and the perceptual, auditory, tactile and kinesthetic sensorial channels." (Radford, 2009, p. 124).

Earlier studies have broached justification from different angles. Some sought to characterize student's arguments and determine whether they accepted a mathematical proof as valid (Stylianides, 2015), while others used the Toulmin model to characterize such arguments (Krummheuer, 2013), focused on teachers' questions and actions that could further explanation or argument (Ingram et al., 2019) or established levels in students' justifications (e.g., Carpenter et al., 2003; Knuth et al., 2009; Lannin, 2005). Some of the open lines of research mentioned in earlier papers included exploring the types of tasks that encourage students to analyze their classmates' generalizations and justifications (Lannin, 2005) and identifying the type of curricular and educational foundations on which to build sophisticated justification (Stephens et al., 2017).

We aim to characterize students' justifications qualitatively, without judging whether they constituted formal mathematical proofs. We deemed justification to be sophisticated in functional contexts when it was precise, explicitly mentioned the variables involved, expressed the relationship between variables bearing in mind a number of mathematical elements and referred to indeterminate quantities. The acknowledged importance of justification in the process of learning mathematics justify our interest in deepening our understanding on how students justify their answers and how this process may be conditioned and mediated by various tasks characteristics. In this case the characteristics consider are the type of justifications demanded by the task and the semiotic systems used in the task. Part of the originality of this study is to analyze the students' answers considering these two components of the tasks. We discuss the differences identified between written and oral justification in terms of the idea of semiotic contraction providing new insight into the development of students' awareness of the functional elements involved in the early algebra tasks proposed as well as the role of the linguistic exchange provoked by the task.

2. Theory and Background

2.1. Justification. In this study, justification is defined as a social process in which mathematical knowledge is explained, verified and systematized based on ideas, definitions and mathematical properties which, like the representations used to express the concept, are within the conceptual reach of the classroom community. Its role depends on the community at issue (Staples et al., 2012). As an educational task, justification may be a means to learn mathematics and solve mathematical problems, enabling students to heighten their understanding of mathematics and improve their mathematical skills.

Justification and argumentation have been defined in a variety of ways in mathematics education literature. According to Chua (2016), justification is a way to determine and explain the truth of a conjecture or assertion. Its roles include determining the truth to dispel one's own doubts or persuading others that a conjecture is true. Similarly, Ayalon and Hershkowitz (2018) and Simon and Blume (1996) stressed that it is both a verbal and a social activity. They positioned justification in a social space as part of classroom discourse for contrasting ideas, reflecting and reasoning. Therefore, it is based on knowledge shared by the community.

Justification tasks may be classified by their nature and purpose or by the element to be furnished in the justification (Chua, 2016). For example, in justification task of elaboration where the purpose is to explain *how*, students are expected to include a description of the method or strategy used to find the result. In task of validation the aim is to explain *why*, with students expected to give reasons or evidence to support or refute a mathematical idea.

2.2. Earlier Research on Justification. The different levels of student justification established in the literature include: non-justification; reference to an authority; empirical evidence; generic example; general argument not constituting an acceptable proof; and deductive justification.

Empirical evidence-mediated justifications are inductive or perceptual, i.e., based on examples rather than on a general relationship (Carpenter et al., 2003). A generic example is a deductive justification expressed in connection with a particular instance (Lannin, 2005). A general argument not constituting an acceptable proof is one that includes non-feasible or mathematically incorrect arguments or an incomplete argument which, if completed, would be acceptable (Knuth et al., 2009).

Blanton (2017) noted that in studies conducted in a functional context, elementary school students often resort to empirical cases when justifying their answers. Similarly, Lannin (2005) concluded that sixth-grade students, when performing generalization tasks in numerical situations, tended to use empirical justification or generic examples. In the latter case generality was identified by the educator rather than the students. Carpenter et al. (2003) stressed that to help students' reason about the context or structure of a problem guiding them toward general arguments is more important than testing specific cases. In a study with sixth- to eighth-grade students, Knuth et al. (2009) observed that example-based justifications prevailed and justification grew more sophisticated with age. Although lacking mathematical rigor, students could prove the general case, perhaps because those students had participated previously in activities that favored justification. Establishing general arguments was the task that posed greatest difficulty.

2.3. Approach of the Semiotic and Social Perspective of Algebraic Thinking. Algebraic thinking can be defined from multiple perspectives (Carraher & Schliemann, 2018; Kieran, 2014). In this study we assume algebra refers to indeterminate quantities (unknowns, variable, parameters or generalized numbers) which are used in an analytical manner. It involves reasoning about generality, recognizing the underlying algebraic structure in a situation and relations between quantities. Algebra as a language for expressing and manipulating generality (Mason et al., 1985) resort to idiosyncratic or specific modes of representation culturally evolved (Radford, 2018).

Concerning the modes of representation in the study of algebraic thinking, we consider thinking is produced in and through a sophisticated semiotic coordination of talk, body, gestures, symbols and tools. It is not just abstract and intangible ideas situated in the mind (Radford, 2009). Signs are the keys to understanding and interpreting how people learn and understand. Signs are psychological tools that enable subjects to reflect and plan actions and act as cultural mediators (Radford & Sabena, 2015). Invoking Vygostkian premises, we deemed signs to be included in children's activity and alter the way they understand the world and themselves. As that transformation depends on the collective social meaning and use of signs, it is related to their historical and cultural role (Presmeg et al., 2016).

The evolution of the meanings of signs is closely related to social interaction because it is the way in which ideas are symbolized and mathematical meanings change with communication and interaction with others. The meaning of signs arises and is materialized and transformed during a singular communicative situation, thanks to the linguistic exchange stablished between the users. In other words, they are developed according to the demands of communication and social interaction (Wertsch, 1985/1995). To understand the meaning of signs, we cannot reduce its interpretation to just what they represent but rather we must understand the type of activity that they allow (Vergel, 2014).

Generalization is a central aspect in algebraic thinking (Kaput, 2008; Mason, 1996) which has been described in several ways in the literature. This activity can be understood as a process (generalizing) as well as a product (generalization) (Ellis, 2007). The process implies: (a) to identify elements common to all cases, (b) to extend reasoning further than the range in which was originated, and (c) to derive more wider results than the particular cases and to provide a direct expression that allows to obtain any term (Ellis, 2007; Strachota et al., 2018). In addition, we think of generalization is constituted by layers which acquire higher sophistication in relation to the semiotic systems used to reason and express generality (Radford, 2010). In general, we share the view that algebraic thinking may be cultivated before algebraic notation is introduced (Carraher & Schliemann, 2018; Radford, 2018).

2.4. Functional Thinking. Research on functional thinking studies functions and families of functions in real-life situations (Cañadas & Molina, 2016). In the context of classroom algebra, a function is a mathematical statement that describes how two quantities covary. It comprises a domain, a target set (codomain or range if constrained to the values adopted by the function) and a rule whereby each element in the domain is paired to a single element in the codomain. The values of the independent variable lie in the domain and those of the dependent variable in the codomain. The definition of which variable is dependent and which independent is conditioned by how the data are presented in the tasks proposed (Blanton et al., 2011).

In this study we propose problems involving linear functions of the forms $y = ax$ or $y = ax + b$, where a, x and b belong to the set of natural numbers. In this context, we consider two of the ways in which the functional relationship may be expressed: directly (how the dependent variable relates to the independent variable) or inversely (how the independent variable relates to the dependent variable).

Functional thinking is present when students establish covariation or correspondence relationships between the variables involved in problems (Smith, 2008). While recurrence relationships may also be established, they are only considered to be functional when they entail the analysis of both variables by establishing a relationship between them.

Functions may be represented in many ways and each representation merely expresses some of its properties. As the starting point for understanding functions, verbal language can be used to formulate a (generally qualitative) description. Tabular representations organize the pairs of elements related by the function and help identify and describe the changes between variables (Blanton, 2008). Symbolic representations afford a general qualitative and quantitative view of the function, from which to abstractly analyze its behavior. The depth of students' understanding of functions depends on how they develop the skill to use a variety of representations and understand their inter-connections (Blanton, 2008).

3. Method

3.1. Participants. This qualitative, exploratory and descriptive study consisted in a CTE (Cobb & Gravemeijer, 2008) with a group of 24 second-grade (7- to 8-year-old) and 25 fourth-grade (9- to 10-year-old) students. This paper discusses only the data for the fourth graders as second graders furnished very few and very vague explanations. Another reason for choosing the older group of students was that they had studied addition, subtraction and the basics of multiplication and division and worked with numbers up to one million. Prior to the working sessions, the students had received no instruction on generalization or expressing algebraic ideas.

They were enrolled in a charter school in southern Spain in a very low-income level neighborhood populated by families at risk of social exclusion. Many of the students were members of socially disadvantaged families of either Romani or Spanish-speaking immigrant origin. We chose to work at such school due to their availability and good disposition to participate.

Our aim is to describe student activity by communicating their ideas and to understand the development of algebraic thinking in a situated way. We assume that their circumstances may involve harder conditions for successful learning. We share the idea that educational phenomena are context sensitive, therefore, we aim that these real references guide future action through reflection (Radford & Sabena, 2015) without pretending to be directly generalizable to other contexts.

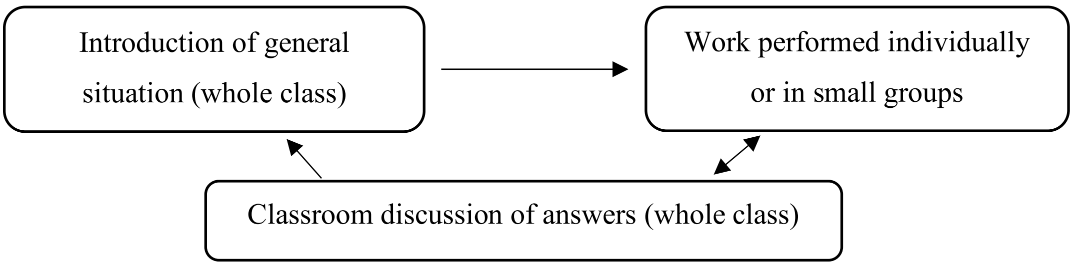
3.2. Instruction Sequence. The design for the fourth graders included an individual questionnaire, two (one initial, one final) individual semi-structured interviews and four 60-minute classroom sessions. The school was visited once a month for 6 months.

Students' normal classroom arrangement in groups of three or four was retained for the working sessions. One researcher played the part of teacher-researcher to monitor the variables defined in the experimental design (Kelly & Lesh, 2000). Her role consisted primarily in encouraging students to participate actively and interact with one another and to clarify their doubts around the tasks. Other team members acted as observers or video-recorded the sessions.

The sessions were divided into three parts (see Figure 4-8). At the beginning of each the teacher-researcher introduced or repeated the general situation that constituted the context for the tasks at hand and ensured that all the students understood them. In the classroom discussions the students were allowed to express their ideas, ask classmates for explanations about theirs or make suggestions to improve the proposed answer. The order in which the parts were conducted was not strictly linear: after a classroom discussion the students could return to their working groups or work on a new task. Whole group discussions and students' interaction were promoted due to the importance of social interaction in the development of algebraic thinking.

Figure 4-8.

Session parts



Contexts and vocabulary were chosen to be familiar to participants. The analysis conducted of each session served as a basis for decisions affecting the ones that followed. Because student communication and activity are closely related to the demands of the task, we include different types of justifications, representations, contexts, and functions in the task design. Task characteristics are listed in Table 4-9.

Table 4-9.

Worksheet tasks

Task characteristic	Session 1	Session 2	Session 3	Session 4
Context	Amusement park		Birthday: tables and boxes	
Function	y = 2x + 1	y = x + 3	y = 2x or y = x + x	
Type of justification				
- Elaboration	✓	✓	✓	
- Validation				✓
Semiotic system				
- Natural language	✓	✓	✓	✓
- Figures or drawings		✓		
- Numerical: numbers only	✓	✓	✓	✓
- Numerical: Arithmetic expressions				✓
- Tabular			✓	✓
- Alphanumeric language				✓

In the tasks involving the function *2x+1* the students encountered problems due to lack of arithmetic knowledge and numerical understanding. As their focus on the operation per se was an obstacle to generalizing and justifying the inter-variable relationship, we designed the following sessions around simpler functions.

The worksheet semiotic systems grew gradually more complex. Sessions 3 and 4 involved the same context and functions but different semiotic systems.

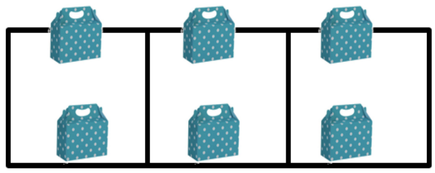
3.3. Data Collection. Data was collected from three sources: worksheets; video recordings with a fixed camera located at the rear of the classroom; and mobile camera video recordings of students as they worked.

This paper discusses the tasks proposed in the last two sessions, described in detail below. We focused on them because the function involved accommodated both multiplication and addition and both the semiotic system and the type of justification were broader than in the earlier sessions. The general situation for the tasks performed in the two sessions is shown in Figure 4-9.

Figure 4-9.

General situation for sessions 3 and 4

Isabel is getting ready for her birthday party. She puts two boxes containing little presents for her guests on each table. She arranges the tables in a row as in the drawing.

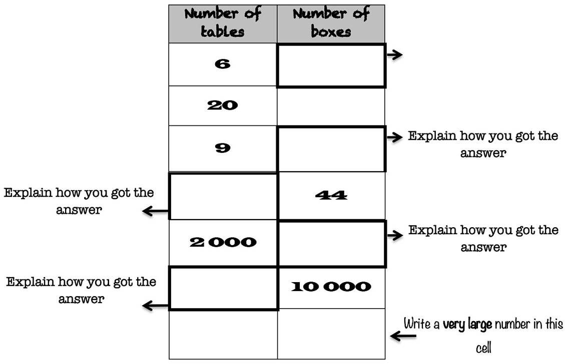


As the numbers of tables and boxes were represented by numbers only in the third session, the functional relationship was implicit, for the objective was for the students to discover it. In the fourth session the functional relationship was sometimes expressed explicitly. Although each task was associated with just one type of justification, in the classroom discussions both could be observed, depending on each student's intention when expressing their opinion.

3.3.1. Description of Session 3. The teacher-researcher introduced the situation with pictures of the tables and boxes, which she pasted on the blackboard until achieving the representation shown as shown in Figure 4-9. In the four specific cases initially described the students identified common elements, established the relationship between the number of tables and number of boxes and tabled the data. They then broke into small groups to perform the tasks shown in Table 4-10. The session ended with a classroom discussion of the answers.

Table 4-10.

Tasks on the session 3 worksheet

<table>
<tr><th>Task</th><th>Case type</th><th>Semiotic system</th><th>Type of justification</th></tr>
<tr><td>Fill in the blank cells in the table with the number of boxes, given the number of tables, or vice-versa. Justify the answer. (Part of the table is reproduced below.)

</td><td>Non-consecutive specific cases</td><td>Numbers only
Tabular</td><td>Elaboration</td></tr>
<tr><td>Answer open questions to generalize the relationship between boxes and tables.
• How do you know how many tables there are when you know the number of boxes?
• How do you know how many boxes there are when you know the number of tables?</td><td>General case</td><td>Natural language</td><td>Elaboration</td></tr>
</table>

3.3.2. Description of Session 4. The fourth session began with a review of the situation introduced in the third and the analysis of a few examples to ensure students remembered the relationship between the numbers of boxes and tables. They then solved the tasks listed in Table 4-11, first in writing and then in a classroom discussion in which each student explained their answer while a classmate stated whether they agreed or otherwise. The session ended with a classroom discussion of the answers to tasks 1 and 3.

Table 4-11.

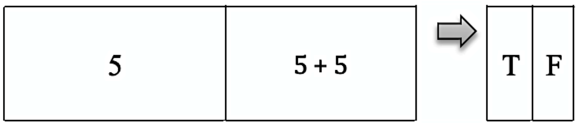
Tasks on the session 4 worksheet

<table>
<tr><th>Task</th><th>Case type</th><th>Semiotic systems</th><th>Type of justification</th></tr>
<tr><td>Task 1
Analyze specific cases in a table and determine whether the relationships are true or false. (Part of the table is reproduced below.)
<table>
<tr><th>Number of tables</th><th>Number of boxes</th><th></th><th></th><th></th><th>Explanation</th></tr>
<tr><td>2</td><td>2</td><td>⇨</td><td>T</td><td>F</td><td></td></tr>
<tr><td>4 x 2</td><td>4</td><td>⇨</td><td>T</td><td>F</td><td></td></tr>
<tr><td>1.000</td><td>Twice 1.000</td><td>⇨</td><td>T</td><td>F</td><td></td></tr>
</table>
</td><td>Non-consecutive specific cases</td><td>Numbers only
Arithmetic expressions
Natural language
Tabular</td><td>Validation</td></tr>
<tr><td>Task 2
Explain in writing how to find the number of boxes:
• When there are 1000 tables
• When there are Q tables</td><td>Specific case
General case</td><td>Natural language
Alphanumeric language</td><td>Validation</td></tr>
<tr><td>Task 3
Analyze statements that explain the number of tables and boxes and justify whether they are true or false. Two of the statements were:
• When Isabel has 11 tables she needs 21 boxes.
• When Isabel has Z tables she needs 2xZ boxes.</td><td>Non-consecutive specific cases
General cases</td><td>Natural language
Numbers only
Arithmetic expressions
Alphanumeric language</td><td>Validation</td></tr>
</table>

3.4. Data Analysis. We analyze how students communicate written and oral justifications. We describe their ways of expressing functional relationships and relate them to the demands of the tasks (determined by the type of justification and the way of representing the variable in the task). We analyzed each explanation in terms of whether it was correct; whether it involved the direct or inverse relationship; the mathematical elements used (counting, addition, multiplication, among others); how the variables were referred to; and whether the expressions were interpreted operationally or structurally. In operational interpretation, expressions are seen in terms of processes for which a result must be found. In structural interpretation, expressions are processed as a single entity with no need for calculation. For instance, in the task illustrated in Figure 4-10, a student exhibiting structural interpretation would answer that the relationship is true because "5 + 5" is twice five and the number of boxes is twice the number of tables. An operational interpretation would contend that it is true because "5 + 5 = 10" and ten is twice five.

Figure 4-10.

Sample task, session 4



We analyzed the classroom discussions on the grounds of the video recordings and respective transcriptions. We characterized students' answers by degree of sophistication of the justifications, based on whether: they explicitly identified and mentioned the variables; explained the mathematical relationship between variables; and expressed themselves in indeterminate or general terms. Table 4-12 lists three examples in descending order of sophistication[16].

Table 4-12.

Examples of characterization of answers

Response	Variables mentioned	Mathematical relationship	Expression of indeterminacy
S_{08}: I said six because if I have to take boxes to however many tables, I add two by two.	Boxes Tables	Adding two by two	However many tables

[16] Students' identities are coded as S_i where i=1 … 25. T-R refers to the teacher-researcher.

Table 4-12.

Examples of characterization of answers

Response	Variables mentioned	Mathematical relationship	Expression of indeterminacy
S_{03}: Six times two, six, for six tables times two for two boxes, then six times two is twelve.	Boxes Tables	Six times two is twelve.	Implicit
S_{17}: By adding. 2000	Implicit	Adding	Implicit

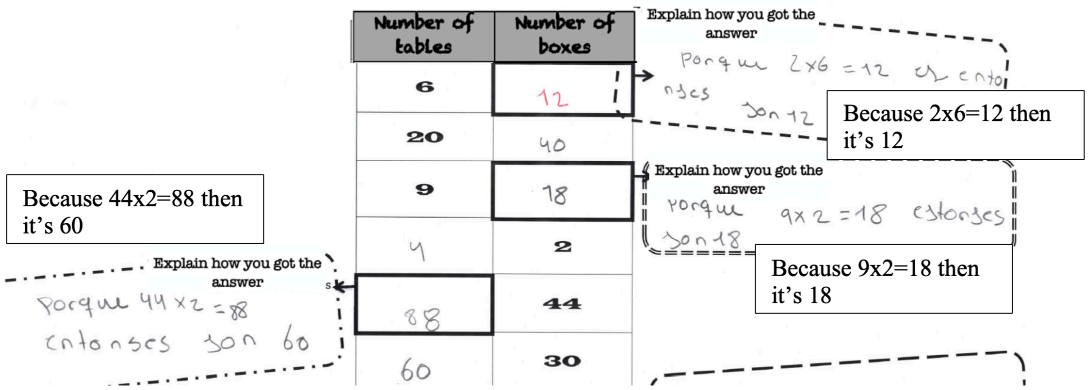
4. Results

4.1. Written Justifications. In the tasks involving *elaboration justification* students preferred to use natural language in brief and at times imprecise answers. Most only mentioned the operation used or whether the relationship was twice or half. The variables were not mentioned.

In the *numerical cases*, some students' justifications invoked the direct relationship, even to explain how to find the number of tables. Their replies included "because I added", "I multiplied", "because it's always twice" or reproduced the multiplication. Figure 4-11 shows S_{09}'s worksheet, by way of example.

Figure 4-11.

S_{09}'s answer

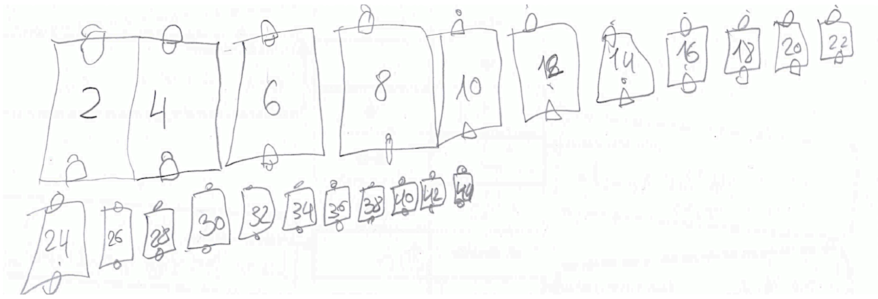


Other students justified the direct and inverse relationships differently, although imprecisely. For the number of boxes, they replied "multiply", "multiply times two", "add" or "find twice" and for the number of tables "subtract twice". Justifying the inverse relationship via subtraction might suggest that students understood adding and subtracting but not multiplying and dividing as inverse operations. They might have also related multiplying times two with adding the same number twice.

Some students also used counting or graphic representations to justify their answers. For instance, when asked to find the number of tables for 44 boxes, S_{07} replied "because I drew the boxes", referring to Figure 4-12.

Figure 4-12.

S_{07}'s drawing, session 3



S_{08} explained how she counted: "because I put two on one finger and I added two by two". She wrote a list in which she established the correspondence between the numbers of tables and the number of boxes (see Figure 4-13). She correctly answered the question involving the inverse relationship, although her justification refers to the direct relationship: "because I found twice [the number]". When justifying the inverse relationship (as he was being video recorded), S_{13} wrote "double 44" and immediately set out to write the sequence two-by-two (see Figure 4-14). When he reached 44, he counted the number of times he counted two-by-two to determine the number of tables. As they based their answers on the numerical sequence, S_{08} and S_{13} seem to have been thinking in terms of the direct relationship, which might explain why, when justifying the inverse relationship, they referred to finding twice rather than half the number.

Figure 4-13.

S_{08}'s count

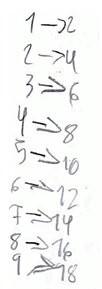
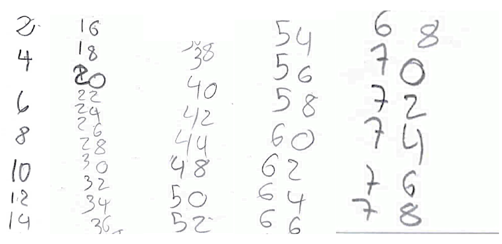


Figure 4-14.

S_{13}'s count



Most students referred only to the mathematical operation applied in their justifications of *indeterminate quantities expressed in natural language*. Some repeated the same justification for the inverse as for the direct relationship, so it is not clear whether they used the inverse relationship. Some students made general statements (e.g. S_{21} said he “multiplied times two” to get the number of boxes and for the number of tables “because in my head I thought it was half”). Others wrote down a number or example that expressed the number of tables and boxes.

Validating justifications were observed to change with the semiotic system used in the task (see Table 4-13 and Table 4-14). Students were asked to justify the false answers only.

Table 4-13.

Answers to task 1, session 4

Semiotic system	Relationship proposed		Student’s reply		No reply
	Tables	Boxes	True	False	
Numerical: numbers	2	2	1	17*	1
	4	8	17*	1	1
Numerical: numbers and arithmetic expression	4 x 2	4	-	17*	2
	13 – 2	13	2	15*	2
	22	22 x 2	12*	5	2
	5	5 + 5	10*	7	2
	10 : 2	10	7*	7	5
Natural language and numerical: numbers	1000	Twice 1000	9*	8	2
	Half of 2 million	2 million	8*	7	4

Note: * = correct answer. Six students were absent.

In the third task (Table 4-14), the statements combined *natural language and another semiotic system,* except in statement 1 which most students mistakenly identified as true. They misconstrued the expression “twice as many” because they failed to heed the order of the variables.

Table 4-14.

Answers to task 3, session 4

Semiotic system	Relationship proposed	Student's reply		No reply
		True	False	
Natural language	1. There are twice as many tables as boxes.	13	5*	1
Natural language and numerical: numbers	2. When Isabel has 11 tables she needs 21 boxes.	1	17*	1
	4. When Isabel has 12 tables she needs 6 boxes.	2	15*	2
Natural language and numerical: numbers and arithmetic expression	3. When Isabel has 4 tables she needs 2x4 boxes.	16*	1	2
Natural language and alphanumeric	5. When Isabel has Z tables she needs 2xZ boxes.	12*	4	2
	6. When Isabel has Z tables she needs Q boxes.	7	8	4
	7. When Isabel has Z tables she needs Z+Z boxes.	8*	7	4
	8. When Isabel has Z tables she needs Z boxes.	3	12*	4

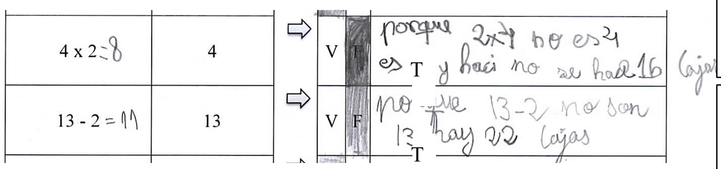
Note: * = correct answer. Statement 6 could be true or false depending on the reasoning.

In the items in Table 4-13 involving *only numbers* and statements 2 and 4 in Table 4-14, most of the justifications mentioned the name of one or both variables and corrected the number of boxes, denoting that the students applied the direct relationship. For instance, in task 1 S_{12} wrote: "it's false because you can't put 2 on two tables. It would be 1 in each 1; 4 is right". As a rule, the calculation or verbalization of the relationship was implicit, except for S_5 who justified the falsehood writing "2 x 2 = 4 tables".

When analyzing *arithmetic expressions*, some students exhibited an operational approach (e.g. see Figure 4-15). Their justifications showed that they solved the operations, applied the direct relationship and then determined whether the results agreed with the numbers proposed.

Figure 4-15.

S_{18}'s justifications, session 4



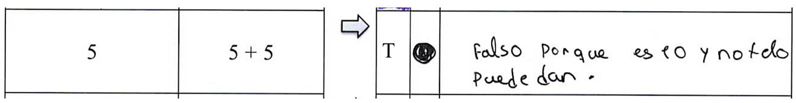
because 2x4 isn't 4, it's 8 and here it doesn't say 16 boxes

because 13-2 isn't 13. There are 22 boxes.

Other students rejected arithmetic expressions as the right answer, possibly because they believed only answers expressed as a number were valid (see Figure 4-16).

Figure 4-16.

S_{09}'s justification in session 4



False because it's 10 and the answer can't give it to you.

In the statements involving *alphanumeric language* to refer to indeterminate quantities, some students assigned values to letters to justify their responses. Focusing on the shape of the letter they sought a number with a similar morphology (e.g. associating 7 and 2 with Z) or they assigned them a value at random or the value beginning with the letter (e.g. zero to Z). After assigning a value, they applied the functional relationship and analyzed the statements. These students did not refer to indeterminate quantities or use the alphanumeric system, although they identified the relationship. Some answered statement 8 generally. S_{17}, for instance, explaining it was false because it couldn't be the same number, had no need to assign the letter any quantity.

4.2. Oral Justifications in Classroom Discussions. Students expressed and communicated the relationship between tables and boxes better orally than in writing. Counting two-by-two was the first mathematic tool that enabled them to explain and verify their answers. Their justifications became more sophisticated as they referred to the relationships *twice* or *half*, with explicit mention of the variables involved.

In session 3, students' *recognition of common elements in some of the cases described* attested to progress toward generalization. In the classroom discussion during the introduction of the general situation around the number of tables for 16 boxes, the students either wielded no or only very general arguments to justify their answers. The discussion ended when S_{13} explained why there were eight tables, basing his justification on counting the number of tables on his fingers. The rest of the class agreed to the answer. Some of the students' answers to other cases are set out in Table 4-15.

Table 4-15.

Answers in the initial discussion, session 3

Question	Reply	Variables explicitly mentioned	Mathematical relationship	Expression of indeterminacy
I have two tables: how many boxes do I need?	S_{18}: four, because I counted two, four.		Counting 2-by-2	
I have 12 boxes: how many tables are there?	S_{08}: I said six because if I have to take boxes to however many tables, I add two by two.	Boxes Tables	Adding 2-by-2	However many tables
	S_{23}: there are six tables because twelve is twice six.	Tables	Twice	
	S_{03}: Six times two, six, for six tables times two for two boxes, then six times two is twelve.	Tables Boxes	Multiply times two	

In connection with generalization involving *reasoning beyond the initial range of values*, the justifications given in the first part of the session 3 reappeared in the post-task classroom discussion. Some students answered and justified correctly while others answered correctly but used the mathematical relationship incorrectly in their arguments. For instance, the question: How many tables are there if we use two boxes? prompted the following dialog:

S_{04}: One, because twice two is one.

T-R: Twice two is one?

S_{03}: No

T-R: Why not, S_{03}?

S_{03}: She said twice two, which is four.

T-R: What's is it then?

S_{03}: It would be half, half of two.

Expressing and discussing their justifications with classmates enabled students to identify their errors and clarified the use of mathematical concepts. The aforementioned situation recurred in the discussion around how to find the number of tables for 44 boxes. S_{08} said by multiplying and S_{21} corrected him saying: "You found half, S_{08}. Not twice. Because if you'd multiplied you'd have 88". S_{04} and S_{08} justified their answers by invoking multiplication, a relationship that had been the object of previous discussion. While they found the right number of tables, they alluded to the wrong mathematical concept when explaining their answers.

When *deriving broader results from specific cases* at the end of session 3, the classroom discussion focused on examples involving indeterminate quantities. Here the teacher's guidance was instrumental to improving justification and generalization. Students were also found to resort to empirical justifications as observed in other studies (e.g. Carpenter et al., 2003). Some of the teacher-researcher's conversations are transcribed below.

T-R: When we calculate 500 million or any number of tables, what do we have to do to get the number of boxes? Can anyone explain that? I mean whichever of those numbers, whichever. What we would have to do to find the number of boxes when I'm given whatever number of tables?

S_{02}: Twice five hundred thousand.

Although asked for an indeterminate number, S_{02} based her reply on a specific case. S_{14} answered the same way.

> *S_{14}:* What I did… since look if 40 is the number of boxes and twenty the number of tables, well what I did, I thought twenty plus twenty is 40 and I added.

Those arguments failed to grasp and express indeterminacy. The students explained the mathematical relationship using specific cases. Whereas S_{02} made no explicit mention of the variables involved, S_{14} referred to the number of tables and boxes and the mathematical operation that linked the two. Therefore, although both arguments were based on empirical cases, S_{14}'s was more sophisticated, given the explicit mention of the variables.

In the following dialog, the teacher-researcher pursued greater precision in the S_{21}'s arguments, affording him the opportunity to express himself more generally

1. *T-R:* How can I find the number of tables if I know how many boxes I have? However many.
2. *S_{21}:* The number of boxes, say, for 500 million tables, no because there always have to be more boxes than tables.
3. *T-R:* But if they tell you how many boxes, what do you do right away to know how many tables?
4. *S_{21}:* Well… half. I'd think half the number of boxes.
5. *T-R:* When you know how many tables, what do you do to know how many boxes?
6. *S_{21}:* Multiplying.
7. *T-R:* Times?
8. *S_{21}:* Times two.
9. *T-R:* What would you multiply times two?
10. *S_{21}:* 500 million times two. 10 million.
11. *T-R:* So, when you know how many tables, what do you do to know how many boxes?
12. *S_{21}:* Multiply.

13. *T-R:* When you know how many tables, what do you do to know how many boxes?
14. *S_{21}:* Multiply times two because there are two boxes on each table.

S_{21}'s arguments referred to the variables indeterminately. Although in lines 2 and 10 he tried to use a specific amount in his explanation, he did not pursue that line of thought but expressed the relationship between tables and boxes in two ways, first qualitatively (line 2) and then mentioning the operations that would yield any number of tables or any number of boxes (lines 4 and 14).

In the following case the indeterminate quantity was represented as "Q". The teacher-researcher said that "Q" represented a number they did not know, that could be any number. The students' first reaction was to ask what that meant and related the letter with the numbers 15 (*quince*) or 40 (*cuarenta*, misspelled as *quarenta*). S_{01} contended:

S_{01}: It depends, because if it's the letter there aren't any tables but if it's the number there are. Then it's double the number.

T-R: If it's the letter there are no tables?

S_{01}: If it's not the letter, it's 40 and there are 80 boxes.

T-R: And how do you know there are 80 boxes?

S_{01}: Because there's twice the number [of tables].

In this case, by accepting that the letter could be a number, the student established the relationship indeterminately, although the variables were implicit in her explanation. Then she validated her answer empirically (assigning Q a value).

5. Discussion

As a general purpose, we sought to determine how justification may be conditioned by the characteristics of the tasks proposed and to describe the development of students' ability to justify. In this regard, we observed that the functional relationship was express in a more sophisticated way when students interacted orally justifying their answers. Whereas students' written justifications omitted information or were imprecise.

This result might be explained by using Vygotsky's view of language and signs as instruments of mediation that are formed accordingly to the demands of communication (Vygotsky, 1934/1995 cited by Wertsch, 1985/1995, p. 97). Vygotsky's ideas about kind of speech serve us to explain the differences detected between oral and written justifications. We relate written justifications with written monologue kind of speech and oral justifications with dialogue kind of speech. The distinction between these kinds of speech is not the amount of people involved but rather the degree of participation between them. According to Vygotsky monologue is a more complex, superior and historically later developed than dialogue. Following Yakubinskii's work (1923 cited by Wertsch, 1985/1995), he considers dialogue as a natural form of interaction and monologue as an artificial late form. Written monologue is more complex due to the requisite of sharing a communicative context with the reader and fully explaining the linguistic formulations (Wertsch, 1985/1995, p. 102). Written justifications lacked additional questions about the comprehensibility of the message or about its adequacy to the demand of the task, so students wrote short messages without questioning whether someone else could understand them or whether they were answering the question. Instead, with classroom discussion students became actively involved in the process. Recognizing classmates' ideas helped them to both adopt more sophisticated solving strategies and verbalize their explanations more precisely. The teacher's intervention proved to be instrumental for introducing generalization in students' justifications and motivate them to refer to indeterminate quantities.

Our results might also be related to the design of the tasks because instances to discuss and reflect on the written productions were not included. Banes et al., (2017) present an experience with fifth-grade students where the importance of thinking about written explanations and motivating students to co-construct them stands out. They show that a collaborative writing lesson with an authentic audience sparks powerful class discussion and engages students in deciding what should be included in a mathematical explanation with different purposes and audiences.

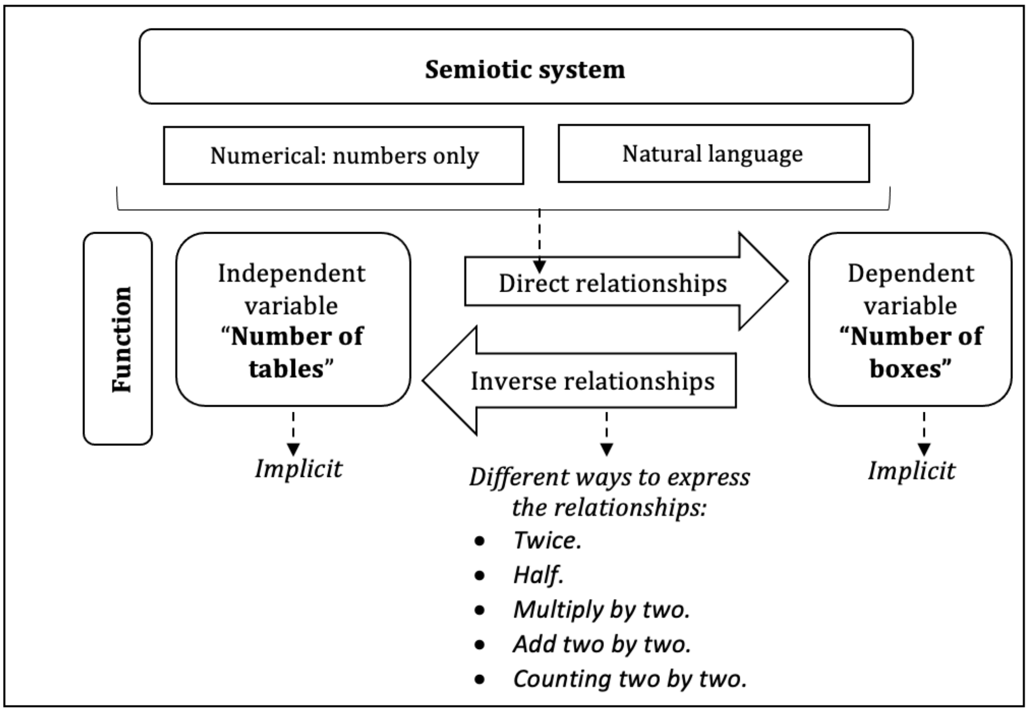
In our study, the results showed that students' oral justifications helped them express the relationship between tables and boxes more precisely (second research question). Possibly including in the design a discussion about the understanding of their written justifications for the inter-variable relationship would have favored the indeterminacy present in these. That supports the importance of socializing answers in functional contexts to help contrast ideas, as observed in studies that analyzed patterns (e.g. Radford, 2018).

Regarding our first research question, we found that the characteristics of students' written justifications depend on the type of justification and semiotic system involved in the task. Accordingly, students focused on one or another element of the function. Students used various semiotic systems to refer to variables and to reason and express the functional relationship, which did not always coincide with the ones used in the task. The students resorted to trusted entities to manipulate and give meaning and expression to the relationships identified.

In *elaborating justifications*, similar written answers were given in tasks involving only numbers and those involving the general case expressed in natural language. Students tended to use natural language in brief and, at times, imprecise answers. Most only mentioned the operation used or the relationship in different ways (see Figure 4-17) and not the variables (boxes and tables). These results show that by asking students for an argument to explain how they obtained the answers, the communicative demand in this task led students to focus more on the functional rule than on the variables, leaving the latter implicit in the questions.

Figure 4-17.

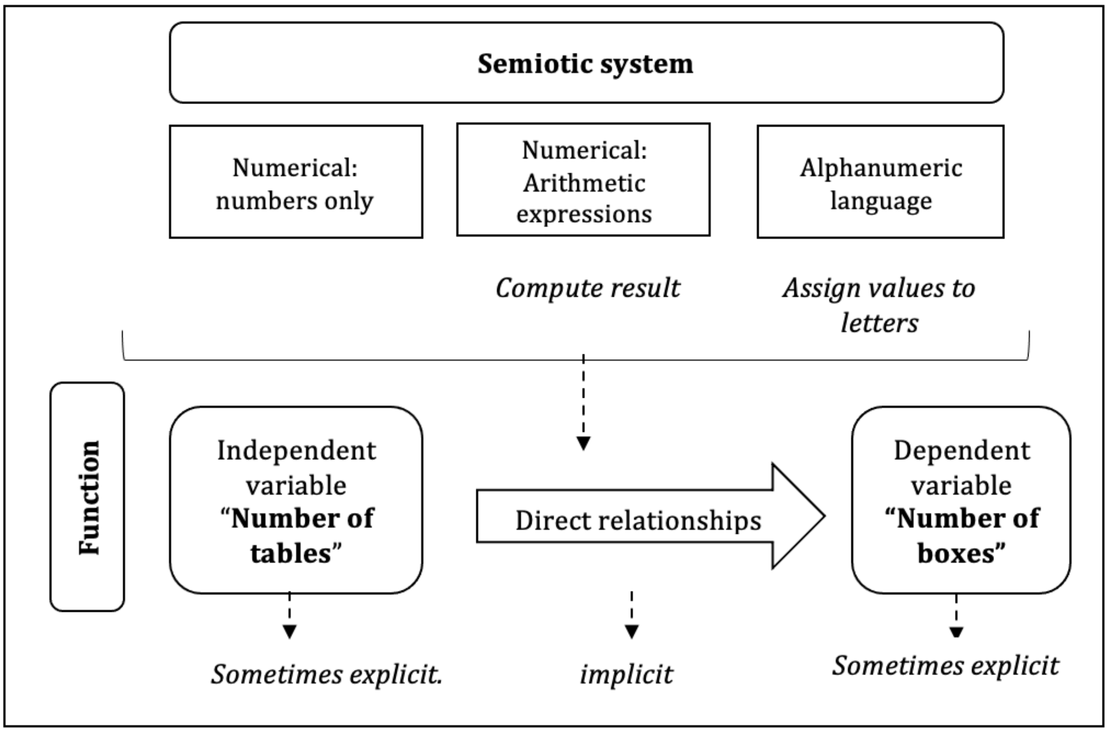
Elements of functional thinking identified in elaborating justifications



In comparison with elaboration justification, the communicative demand of *validation justification* was different, so students focused on other elements of the function. The elements explicit and implicit in their explanations differed (see Fig.11). False statements were explained differently depending on the semiotic system involved. More students mentioned the names of the variables in the statements involving numbers only. Mainly, students applied the direct relationship to check whether the relationship between the quantities is correct or not. They made the number of boxes to always be the double of the number of tables but did not make the functional relationship explicit in their justifications (see Figure 4-18). Therefore, our results suggest that students interpreted the elaborating justifications as demands to clarify the processes applied while validating justifications led them to refer to pairs of concrete values within a context.

Figure 4-18.

Elements of functional thinking identified in validating justifications



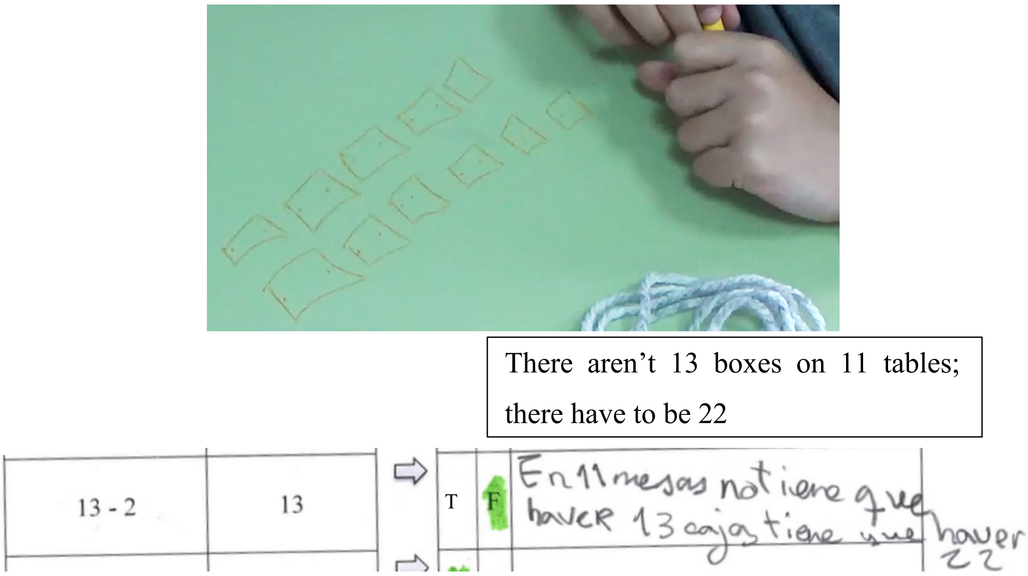
The arithmetic expressions were broached operationally. The students failed to accept as valid equivalent expressions and performed the calculations to validate the equivalence of the result. In the third session students recognized that the functional relationship could be expressed as an addition or multiplication or *twice* and identified the equivalence. When equivalent arithmetic expressions involving those ideas were presented in the fourth session, however, they found it difficult to make the connection. No structural interpretations of the expressions were observed. Structural approaches are identified as a scarce but a natural consequence of arithmetic learning (Knuth et al., 2005) which become more frequent when they are explicitly promoted in instruction (Molina & Mason, 2009). However, previous studies focus in arithmetic expressions within the approach to algebra named "Generalized arithmetic". There is no information about how accessible is the structural interpretation of expressions in a functional context. Our results suggest this might not be an approach to algebra that naturally favors this structural view. Therefore, proposing tasks that include such representations and discussion about them is required to foster the generalization of arithmetic relationships. Working with the structure of numerical expressions in elementary school should help students understand secondary school algebra (Kieran, 1989, 2018). The inference is that the functional approach to algebra that guided this study is related to another conception of algebra: generalized arithmetic.

When working with alphanumeric language students first assigned the letter a value and then applied the functional relationship. Their written answers furnished no evidence of an indeterminate interpretation of letters. Although their explanations denoted an understanding of the situation, students' communication skills in that context were underdeveloped. We must say that these results were expected as it was the first time that students were asked to use letters to refer to unknown quantities. The first students' reactions in our study match those described in previous studies. For example, Küchemann (1981) mentions that students have difficulty interpreting the letter as a varying quantity. Also, letters tend to be interpreted as concrete objects, ignored or related to a numerical value. Different meanings would affect how activities can be solved. Booth's research (1988) identifies possible explanations for the origin of errors in the use of letters. One of these is the focus of algebraic activity and the nature of its responses. It points out that while in arithmetic the aim is to find a numerical answer, in algebra the focus is on processes, relationships and their expression in general. As a consequence of not identifying the differences in the type of answer, in algebraic contexts students expect to write a numerical answer. Other studies describe how students manage to interpret letters as variables that represent indeterminate quantities in a context of functional problem solving (e.g. Ayala-Altamirano & Molina, 2020; Blanton et al., 2017; Brizuela et al., 2015). However, this requires more time than just a pair of sessions.

Arithmetic and Alphanumeric expressions were the ones that posed the greatest challenge to students. We observed in the recordings of students answering their worksheets that their gestures and drawings made situations more specific, favoring reflection and concentration on mathematical ideas. For example, S_{13}, after defining the “13-2 tables” / “13 boxes” relationship as true, was heard to say, while pointing at the numbers, “11 tables and two boxes, 13”. S_{01} told him the statement was false, that there would be 22 boxes. S_{13} then looked at the drawing of the tables shown at the beginning of the session, with S_{01} drew the tables shown in Figure 4-19 and concluded that the number of boxes proposed was wrong.

Figure 4-19.

S_{13}'s representation and S_{01}'s written reply



S_{12} was recorded during session 3 as she replied to statement 6. First, she drew 27 lines, then counted them one-by-one, but touching the top and bottom of each line. Those movements would represent the number of boxes per table. Halfway through the count, she stopped and decided that wasn’t the solution, writing “it’s false because Z is 2 and Q is 15[17] and that’s not it”. Her gestures showed that she understood the situation and could apply it to specific cases by counting. She realized that there were two boxes per table but was unable to express the idea in general terms.

Although in this paper we do not aim to describe students’ gestures in detail, we turn to them when needed due to the importance they had to give sense to tasks. Mason (2017) points out that students turn to entities of trust to manipulate and give sense to the identified relations. In a cyclic process of representations becoming more sophisticated,

[17] The Spanish word for fifteen is ‘quince’, hence the ‘Q’.

students keep adding layers of appreciation, comprehension and understanding. In a similar way Radford (2010) explains how algebraic thinking is composed by layers of understanding that depend on the semiotic systems involved.

6. Conclusions

This study contributes to a fuller characterization of students' thinking when facing situations involving functions. We contribute to the study of functional thinking from a semiotic perspective inspired by some of Radford's papers (2010, 2018). Promoting justification in the classroom led to generating instances of communication in which we identified how students thought and expressed themselves through signs. Specifically, by means of semiotic systems we could observe how students thought and talked about functions. We would like to reiterate that the situations we describe in this study are context sensitive, so we do not intend to generalize directly to other scenarios. To guide future action, reflection is fundamental (Radford & Sabena, 2015).

The *twice* and *half* relationships were pivotal factors in the sessions analyzed. While those concepts can be used to perform tasks involving a known relationship without understanding the function per se, in this study they also proved useful for prompting discussion on the relationship between two quantities that vary in a joint way, even if the students do not explicitly understand this as a functional relationship. Our findings emphasize the importance of classroom discussion, which was furthered by posing questions that induced different types of justification of how to reach a certain answer or why certain relationships or statements were false. The justifications recorded helped describe students' thinking by analyzing the semiotic systems used or their interpretation of the systems suggested in the tasks.

Although the oral justifications expressed in classroom discussion were more sophisticated than those furnished in writing, both approaches are important. The written justifications helped us understand the consistency between what students said and did. Whereas in classroom discussions many correctly identified *twice* as the relationship involved, their written answers failed to detect the cases where its application was in order.

When something was difficult to understand, the existence of a variety of systems enabled students to broach the situation with one that was familiar. Other studies address tasks in which variables were represented either numerically only or with letters only. Part of the originality of this study is the use of a variety of systems and arithmetic expressions.

By proposing tasks with non-consecutive particular cases, students' establishment of correspondence relations was favored. This contrast with results from previous studies (Carraher & Schliemann, 2007) that show predominance of use of recursive relations, a separated attention to variables and higher difficulty to identify how the variables covary, when students are asked about consecutive cases.

One recommendation for designing lessons would be to analyze arithmetic expressions as a way of prompting discussion that would favor a structural interpretation of situations. That approach has been applied in other studies, working with algebra as generalized arithmetic (Molina & Mason, 2009; Blanton et al., 2018). Although we only reported on functional relationship $y = 2x$, our proposal demonstrates the utility of analyzing arithmetic expressions in functional contexts, an approach that also accommodates comparing and studying the characteristics of functions. An open line of research would be to investigate what happens in contexts involving other types of functions.

Different semiotic systems were introduced gradually. Students first analyzed numerical cases and identified the functional relationship. In the following session, depending on their responses, we introduced statements or relationships that used other semiotic systems to express relationships already recognized.

The discussion on the veracity of statements also enabled students to analyze the variables more fully. In an earlier study (e.g. Ayala-Altamirano & Molina, 2020) we showed this type of questions to help students refer to indeterminate quantities and make sense of the alphanumeric system. Here we took the study of functions one step further by introducing other semiotic systems and affording students more opportunities to justify their answers.

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