THE ROLE OF CONCEPTIONS IN ARGUMENTATION AND PROOF

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In this article an analysis concerning the role of students' conceptions in solving a geometrical problem is presented. Even if conceptions do not usually appear in the final proof, they strongly affect the argumentation activity. The main aim of this paper is to show this influence. In particular, through the use of Toulmin's model, we show how conceptions can affect the modal qualifier and the rebuttal of argumentation.

INTRODUCTION

As highlighted by Balacheff (2009) mathematical knowing and proving cannot be separated. Engaged in mathematical problem-solving, learners proceed based on their understanding of mathematical concepts and related process. This activity is a tangle of intuitions, know-how, knowledge and a variety of mental constructs, which allow learners to make choices and take decisions. Proving activity is strictly connected to the on-going argumentation activity involved in solving a problem (Boero, Garuti and Mariotti, 1996). When students construct their argumentations in order to construct a proof they use their conceptions (Balacheff, 2009) that are at the basis of argumentation activity (Pedemonte, 2008) even if in the proof (considered as final product in the proving activity) they are usually not present.

As stated from Balacheff (2009) a conception is not a kind of property or state of knowledge ascribed to a learner, but a property or state of knowledge of a learner in a situation; a conception is a situated mental construct. Furthermore for a given piece of mathematical concept, a learner may not have one conception, but a set of conceptions likely to be mobilized depending on the situations in which he or she is involved. A deeply analysis explaining how conceptions affect the construction of an argumentation is important to compare knowledge used in the argumentation with theorems used in the proof.

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Some previous researches (Pedemonte, 2005) have shown that, if the conception corresponds to the application of a mathematical rule there will probably be continuity between argumentation and proof because this rule can be replaced in the proof with a theorem. Obviously, if the conception is not correct, it cannot be replaced by a theorem in the proof phase. In this case, three possibilities can be identified: the proof is not constructed by the student because he is not able to replace the conception by a theorem; an "incorrect proof" is constructed and it is based on the conception used in the argumentation; the argumentation based on the conception is abandoned and another argument is constructed. As a consequence, it is very important the kind of conceptions mobilized in the argumentation because the construction of the proof strictly depends from it.

The present paper can be considered as a continuation of the previous study in that it analyses the role of the qualifier and the rebuttal in the argumentation. Through the use of Toulmin's model, we show how conceptions intervene in the construction of argumentation when students solve a geometrical problem. In particular, the Toulmin's model allows us to show where conceptions intervene in the students' argumentation (as warrant, backing, rebuttal or qualifier). It is quite obvious that conceptions affect the warrant and the backing in the argumentation (Pedemonte, 2005) if we assume that conceptions lead the construction of the argumentation. On the contrary, concerning the qualifier and the rebuttal some clarifications occur. Indeed, as highlighted by research (Inglis, Mejía-Ramos and Simpson, 2007) the modal qualifier and the rebuttal have an important role in the argumentation activity. They show that it is important to learn to pair intuitive arguments with appropriate modal qualifiers and rebuttals. This is crucial in the process of solving the problem and in particular to produce an appropriate proof.

The aim of this paper is to show that the strength of an argument strongly depends from the mobilized conceptions and the rebuttal can be developed as a related consequence.

In the following sections, after a brief presentation about Toulmin's model, we analyze two students copies taken from a set of data collected from a teaching experiment. This analysis shows how conceptions can affect the modal qualifier and the rebuttal of students' argumentations.

TOULMIN'S MODEL

Toulmin's model (1958/1993) has been used by several researchers in mathematics education (Lavy, 2006; Stephan and Rasmussen, 2002; Hollebrands, Conner, Smith, 2010; Knipping, 2008) to examine students' mathematical argumentations. In this report, Toulmin's model is used to analyze student's argumentations (Pedemonte 2007, 2008).

In any argumentation the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin's terminology the standpoint is called the claim. The second step consists of the production of data supporting the claim. The warrant provides the justification for using the data conceived as a support for the data-claim relationships. The warrant can be expressed as a principle, a rule and it acts as a bridge between the data and the claim. This is the base structure of an argument, but auxiliary elements may be necessary to describe an argumentation. Toulmin describes three of them: the qualifier, the rebuttal and the backing. The force of the warrant would be weakened if there were exceptions to the rule, in that case conditions of exceptions or rebuttal should be inserted. The claim must be weakened by means of a qualifier. A backing is required if the authority of the warrant is not accepted straight away.

Then, Toulmin's model of argumentation contains six related elements as showed in the Figure 1.

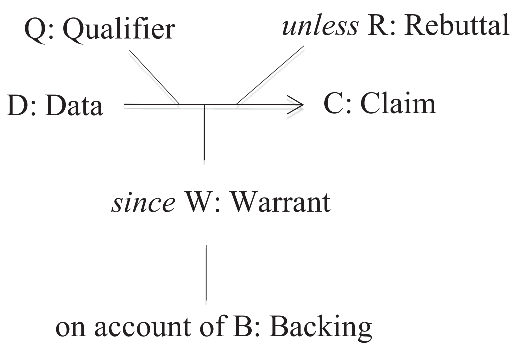
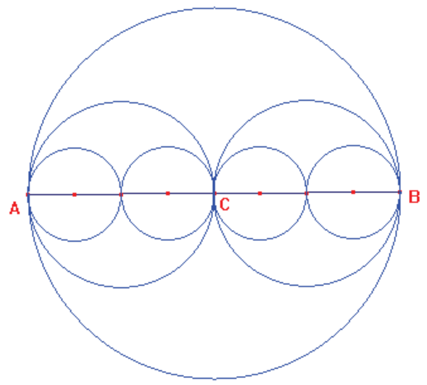


Figure 1: Toulmin's model

EXPERIMENTAL DESIGN

The following examples are taken from a set of data collected in two classes – 11th and 12th grade– in Italy, and in one 12th grade class in France. In total we have analyzed 16 students. The students worked in pairs on a computer running the Cabri-Geometry software. The experiment lasted about an hour and a half. The problem proposed was the following:

Construct a circle with AB as a diameter. Split AB in two equal parts, AC and CB. Then construct the two circles of diameter AC and CB... and so on. How does the perimeter vary at each stage? How does the area vary?



To solve the problem students usually calculate the perimeter and the areas of each curve for the first three or four stages. They consequently observe that the perimeter is constant in each stage while the area is each times the half of the precedent.

In this paper we analyze how conceptions of students lead the solution of the problem and the construction of the proof. To solve the problem students have to establish a relationship between two different settings (Douady, 1985): the spatio-graphic setting (where they see how the radius change from a stage to the following one) and the algebraic setting (where they manipulate formulas). To accept an argumentation as valid the conception should provide the means to account for the coherency between the two settings. Only in this case the modal qualifier of the argumentation is strong enough to allow students to consider the construction of a proof; although in some cases it could not be enough (see Example 1). Furthermore, because students worked in pair, it was easier for us to observe the interactions of different conceptions. Indeed, when students construct different and contrasting argumentations (see Example 2) they are obliged to make a choice to construct a proof.

In the following section we present two examples to show how the students' conceptions affect their argumentations:

- Example 1: the conception leads the student to consider a rebuttal in its own argumentation (rebuttal inside the argument).

- Example 2: the conception mobilized by a student becomes a rebuttal in the classmate's argumentation (rebuttal outside the argument).

The solution protocols are based on transcriptions of audio recordings and on the students' written productions. The assertions produced by students were selected and the argumentative steps were reconstructed. The indices identify each argumentative step. The student's text is in the left column while comments and analyses are reported in the right column. The texts have been translated from Italian/French into English.

Example 1

Nicola and Stefano attend the 12th grade in an Italian school. Nicola is the student who seems to have a "leader role" in solving the problem. He talked much more than his classmate Stefano. At this moment students are trying to understand how change the area of the curve at the second stage (constituted from the two circles) in respect to the area of the first curve (the circle).

6. N: …the area is πr^2, so 7. S: the area 8. N: here it becomes the half of the previous one… the sum of these two circles (*Nicola shows the second curve*) is the half to the first one, the sum of the four circles (*he shows the third curve*) is the half to the second one … it is the radius…yes, the area is the half because we have cut the diameter of the first curve in two parts and consequently the diameters of the two curves are the half of the first curve's diameter… and so the area of the two curves is the half of the area of the first curve. 9. S: yes because 10. N: because the sum of the two little radii is equal to the radius of the first circle, and so the area of these two circles here, is the half of the area of the first circle. 11. S: Ok, we can write…	*Students know the formula of the area of the circle and Nicola understands the relationship between the radius and the number of circles: the radius of one circle of the second curve is the half of the radius of the first curve.* D_1: the area of the first curve is πr^2 → C_1: the area of the second curve is the half of the area of the first curve W: The radius of the second curve is the half of the radius of the first curve Formula of area B: Spatio-graphic and algebraic settings *Note that the students reasoning "lives" in both settings: the algebraic and the spatio-graphic ones.*

Even though both students solved the problem, and they have all elements necessary to construct proof, they decided to measure the areas. The qualifier of the argument is not strong enough to pass to the construction of the proof. Students need to measure the areas to be sure about their reasoning. There is an implicit rebuttal in the student argument; the calculus of the computation should confirm students reasoning.

16. N: wait, wait a moment, we have to measure the areas to be sure… measure the radius and calculate the areas *The students measure the radius and they calculate the areas of the circles using the specific tool in Cabri-Geometry.*	R: unless the calculus obtained measuring areas provide different results Q: Probably D_1: the area of the first curve is πr^2 → C_1: the area of the second curve is the half of the area of the first curve W: The radius of the second curve is the half of the radius of the first curve Formula of area B: Spatio-graphic, algebraic and arithmetic settings *In this case the reasoning is performed in the symbolic arithmetic setting. Nicola needs to check his argument to be sure it is correct. The arithmetic setting represents a "sure" domain that could strength the qualifier of the argument.*

Only when the calculus of the computation confirm theirs reasoning the two students decide to construct a proof. It is interesting to observe that here the reasoning lives in three different settings: algebra, spatio-graphic (drawings), and symbolic arithmetic. Only after measuring the area Nicola is satisfied with his reasoning.

The Rebuttal R is developed from the students (calculus obtained measuring areas have to be compatible with the algebraic results) because if the results of the computation do not provide the results students are waiting for, the argument could be false.

For the perimeter their reasoning is quite similar: they see that in the three cases the perimeter is the same and they verify calculating it with the measurement tool of Cabri-Geometry.

Example 2

Vincent and Ludovic are two 12th grade French students. They promptly see that radius in each curve is divided by two, so using the perimeter and area formulas, they construct the conjecture: the perimeter is always the same and the area of each curve is the half of the area of the previous curve.

9. V: The perimeter is $2\pi r$ and the area is πr square 10. L: yes 11. V: but how does the radius evolve? *r* is divided by two 12. L: yes, the first perimeter is $2\pi r$ and the second is $2\pi r$ over 2 plus $2\pi r$ over 2 … then it is the same 13. V: then yes… 14. L: and it is always the same… look at that... we name *r* the first, *r* is the radius of the first curve, the perimeter of the first circle is…. 15. V: $2\pi r$ 16. L: $2\pi r$ and the sum of the perimeter for the second curve is $2\pi r$ over 2 17. V: plus $2\pi r$ over two, that means $2\pi r$… and so on. The next one is $2\pi r$ over 4 but for 4 times 18. L: then the sum is always $2\pi r$ 19. V: it is always the same perimeter….	*The students consider the radius for each curve and they note that it is always divided by two. They construct the conjecture* D_1: the perimeter of the first curve is $2\pi r$ → C_1: the perimeter of the second curve is $2\pi r$ W: The radius is divided by 2 for each subdivision but the number of circles is double Formula of perimeter B: Spatio-graphic and algebraic settings *As in the previous example, the backing includes the Spatio-graphic and the algebraic settings. However, it is interesting to observe that the reasoning is probably developed observing the drawing but it is generalized in the algebraic setting.*

	D_2: C_1 → C_2: the perimeter of the third curve is $2\pi r$ W: The radius is divided by 2 for each subdivision but the number of circles is double Formula of perimeter B: Spatio-graphic and algebraic settings *Students see that the radius is divided by two for each subdivision (not only for the first curve to the second one).* *Then students generalize their results and construct the conjecture.* D_3: $D_1 \to C_1$, $D_2 \to C_2$ → C_3: the perimeter is always 2πr W: Generalization on the process

In the same way the students consider the areas and they construct their conjecture.

20. L: on the contrary the area is πr square 21. V: in this case … 22. L: hem…. It is divided by two… 23. V: yes, πr over two at the second power plus πr over two at the second power is equal… 24. L: is equal to … πr at the second power over two 25. V: yes, if we divide by two 26. L: yes, the area is always the half of the previous one *Vincent write the area for the third curve and he observe that it is the half of the pre-*	D_4: the area of the first curve is πr^2 → C_4: the area of the second curve is $\pi r^2/2$ W: The radius is divided by 2 for each subdivision but the number of circles is double Formula of area B: Spatio-graphic and algebraic settings

vious one 31. V: the area is divided by two each time….	D_5: $D_4 \rightarrow C_4$ → C_5: the area is divided by 2 each time W: Generalization on the process

The students conjectured that the perimeter is constant while the area decreases to zero. With no hesitation both students solved the problem, and until now it seems that both of their conjectures were validated in the two settings: the algebraic setting and the spatio-graphic setting.

Actually this was probably not the case. Let's see the following part where students decide to consider the limit case.

It is interesting to observe that Vincent and Ludovic have two different conceptions with respect to the limit case. The Vincent's conception is mobilized in the spatio-graphic setting (drawing): Vincent "sees" the perimeter becoming the diameter of the circle. On the contrary, Ludovic' conception is mobilized in the algebraic setting: at the limit the perimeter is $2\pi r$.

37. V: yes, then the perimeter? 38. L: non, the perimeter is always the same 39. V: but in the worst case, the perimeter becomes twice the segment 40. L: what? 41. V: It falls in the segment... the circle are so small 42. L: Hmm... but it is always $2\pi r$ 43. V: Yes, but when the area tends to 0 it will be almost equal... 44. L: no, I do not think so 45. V: If the area tends to 0, then the perimeter also... I don't know … 46. L: I will finish writing the proof. *Silence… Ludovic writes the first proof and he does not pay attention to the Ludovic idea.*	*The two students produce two different arguments.* *Ludovic argument can be represented as follows:* D_{6L}: the perimeter is always the same → C_{6L}: in the limit case the perimeter is always the same W: Generalization on the process B: Algebraic settings *W is constructed by Ludovic observing the regularity of the results at each stage: if for each curve the result is 2πr then at the limit case the result should be the same The Ludovic reasoning is in Algebra.* *For Vincent the limit case is the diameter because he sees the drawing:* C_{6V}: at the limit case the perimeter is the diameter

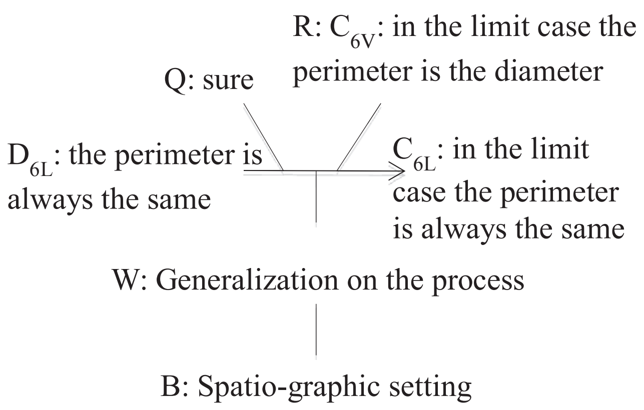
The two students don't agree to each other; they have two different conceptions that lead them to two different claims. Vincent's statement runs as a rebuttal in the Ludovic argument. Nevertheless, it is not strong enough to modify the argument. Indeed, while Vincent's conception is based on perceptive aspects, Ludovic argument is based on Algebra.

Vincent is not really convinced that at the limit case the perimeter is also the same.

47. V: But if the area is close to 0 at the limit case, also the perimeter should be close to twice… to the diameter of the first curve

48. L: it is different, the perimeter is constant

49. V: ah ok…



Although Vincent and Ludovic collaborate well and seem to share the mathematics involved, the types of reasoning they develop on their problem-solving activity differ. Ludovic is working in the algebraic setting, his reasoning is provided by his verification of the correctness of the symbolic manipulation and his knowledge of elementary algebra. For Vincent the reasoning comes from a constant confrontation between what the formula "tells" and what is displayed in the drawings. Both students understood the initial situation in the "same" way, both syntactically manipulated the symbolic representations (i.e., the formulas of the perimeter and of the area), but their reasoning were different, revealing that the conceptions they mobilized were also significantly different.

CONCLUSION

In this paper we have analyzed the role of students' conceptions (Balacheff, 2000) in the construction of a proof. The use of Toulmin's model highlighted some important aspects of this relationship. First of all, we have observed that the conception strongly affect the modal qualifier of an argumentation.

Furthermore, we have observed that the rebuttal can have two different roles: it can be developed inside the argumentation, if it is developed from the arguer himself, or it can be outside the argumentation if it is developed from someone else.

In the first case (Example 1) the rebuttal is constructed to strength the argumentation. If the rebuttal is rejected the conception is validated in another representation system making the qualifier of the argument stronger than before.

In the second case, the rebuttal is constructed because the two conceptions are in opposition. In Example 2, when students consider the limit case, we observed that the two students' conceptions are based on different settings. Algebra is stronger in respect to the spatio-graphic setting, this is why the rebuttal is rejected and the argumentation can be transformed into a proof in a mathematical sense.

REFERENCES

Balacheff, N. (2000). A modelling challenge: Untangling learners' knowing. In *L'apprentissage, une approche transdisciplinaire* (pp. 7-16). Orsay, France: Institut des Sciences Cognitives et de la Communication. Retrieved from http://telearn.archives-ouvertes.fr/docs/00/19/02/92/PDF/Balacheff2000.pdf

Balacheff, N. (2009). Bridging knowing and proving in mathematics: A didactical perspective. In G. Hanna, H.N. Jahnke and H. Pulte (Eds.), *Explanation and proof in mathematics. Philosophical and educational perspectives* (pp. 115-135). New York, USA: Springer.

Boero, P., Garuti, R. and Mariotti, M.A. (1996). Some dynamic mental processes underlying producing and proving conjectures. In L. Puig and Á. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (vol. II, pp. 121-128).Valencia, Spain: University of Valencia.

Douady, R. (1985). The interplay between the different settings, tool-object dialectic in the extension of mathematical ability. In L. Streefland (Ed.), *Proceedings of the 9th International Conference for the Psychology of Mathematics Education* (vol. II, pp. 33-52). Utrecht, Germany: State University of Utrecht.

Hollebrands, K., Conner, A. and Smith, R.C. (2010). The nature of arguments provided by college geometry students with access to technology while solving problems. *Journal for Research in Mathematics Education, 41*(4), 324-350.

Inglis, M., Mejía-Ramos, J. P. and Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*(1), 3-21.

Knipping, C. (2008). A method for revealing structures of argumentation in classroom proving processes. *ZDM*, *40*(3), 427-441.

Lavy, I. (2006). A case study of different types of arguments emerging from explorations in an interactive computerized environment. *Journal of Mathematical Behavior, 25*(2), 153-169.

Pedemonte, B. (2005). Quelques outils pour l'analyse cognitive du rapport entre argumentation et démonstration. *Recherche en Didactique des Mathématiques*, *25*(3), 313-348.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*(1), 23-41.

Pedemonte, B. (2008). Argumentation and algebraic proof. *ZDM*, *40*(3), 385-400.

Stephan, M. and Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *Journal of Mathematical Behavior*, *21*(4), 459-490.

Toulmin, S.E. (1993). *The uses of arguments*. Cambridge, UK: University Press (First published, 1958).