Integration of didactical knowledge and mathematical content knowledge in pre-service teacher training

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We suggest some ideas about how content knowledge and didactical knowledge could be integrated in pre-service secondary mathematics teachers' training. We identify the activities we expect a teacher to perform when planning a lesson and determine the didactical knowledge that she has to put into play in order to do so. We then show the relationship between that didactical knowledge and the corresponding content knowledge.

Introduction

There is an increasing awareness about the importance of teachers' mathematical knowledge as a factor affecting the quality of teaching (Tirosh & Graeber, 2003). Some proposals focus on the teachers' university training and its relationship to school mathematics (Cuoco, 1998). However, research has not shown a clear relationship between teachers' university mathematics knowledge and their teaching (Ball, 1991). Learning the mathematics they are going to teach does not necessarily help, either (Ball, 1988). It seems nowadays that the issue is more concerned with "learning school mathematics in a more 'deep, vast, and thorough manner' (Ma, 1999, p. 120)" (Tirosh & Graeber, 2003, p. 667). But, what this more "deep, vast and thorough manner" is? One possibility is to treat mathematics in general, and school mathematics in particular, from a pluralistic perspective. This

> *pluralistic approach to teaching of mathematics includes the presentation of mathematics as an intuitive subject in which students use pattern recognition to discover mathematical concepts and generalisations, as an empirical subject in which students' investigations give rise to mathematical concepts and generalisations, and as a formalised system of logical consequences. (Cooney & Wiegel, p. 808)*

We want to enlarge the possibilities of this pluralistic view of school mathematics with a series of conceptual and methodological tools that can enable the teacher to reveal the multiple meanings of the subject matter.

In this sense, we want to tackle some of the issues proposed for this discussion group. In particular, we will develop some ideas concerning:

- the competencies that are required for planning secondary mathematics lessons,
- the didactical knowledge that we consider relevant in pre-service secondary mathematics teacher training, and
- the knowledge about the mathematics subject matter that should support and integrate with that didactical knowledge.

We assume a functional approach to didactical knowledge similar to that of Simon's (1995): we ask ourselves about the activities that the teacher has to perform in order to promote her students' learning. Based on the requirements that emerge from those activities, we identify the competencies and knowledge required to perform them. We can then characterize the didactical knowledge that we consider relevant in our pre-service teacher training programs. We will focus our attention on the activities necessary for planning a reduced number of lessons about a specific mathematical subject matter. Since we will refer mainly to the activities concerning this kind of local curriculum design, we will not elaborate on the pedagogical skills required to put the design into practice.

We want to show the relationship between the didactical knowledge and the mathematical content knowledge that the teacher requires to perform her job. We see the didactical knowledge as the knowledge that enables the teacher to reveal the multiple meanings of the mathematical subject matter. These meanings should serve as the groundwork for producing lessons' planning that can enhance students' learning. In this sense, these are the meanings of school mathematics that we consider relevant in its teaching and learning.

In the first section, we describe the didactical analysis that we expect the teacher to carry out when planning one or more lessons. Next, we identify, as didactical knowledge, the mathematics education notions that we consider relevant for performing the didactical analysis. Finally, we suggest how pre-service secondary mathematics teachers training programs should articulate this didactical knowledge with the content knowledge developed in their mathematics training.

Didactical analysis

In this section, we describe briefly the activities that a teacher should perform for planning one or more mathematics lessons. We assume that the teacher has some knowledge concerning the epistemological, cognitive and instructional aspects of school mathematics and that she has explicitly adopted a stand concerning mathematical knowledge in school and its teaching and learning in class. We also assume that she has to deal with a specific mathematical subject matter, usually concerning a particular concept that, for the educational level at hand, requires four to six lessons of teaching and learning.

We structure the above-mentioned activities following the four dimensions of curriculum: cultural/conceptual, cognitive, ethical/developmental and social (Rico, 1997). According to these dimensions, a cycle of the didactical analysis that we expect the teacher to perform can be organized around four groups of tasks, each one corresponding to one dimension: content, cognitive, instruction, and performance. The purpose of these tasks or analyses is to allow the teacher to examine and describe the complexity and multiple meanings of the subject matter in order to design, implement, and assess teaching/learning activities.

Any cycle of the didactical analysis begins with the identification of the student's knowledge for the subject matter at hand (Gómez & Rico, 2002, see Figure 1). We expect the teacher to use her knowledge and previous experience for establishing the tasks that the students can and cannot solve, the type of activities her students can and cannot get involved in, the mistakes they can make, and the difficulties underlying those mistakes. With this information, and taking into account the global planning of her course, we expect the teacher to determine the goals she wants to achieve and the mathematics content she wants to work on (box 1).

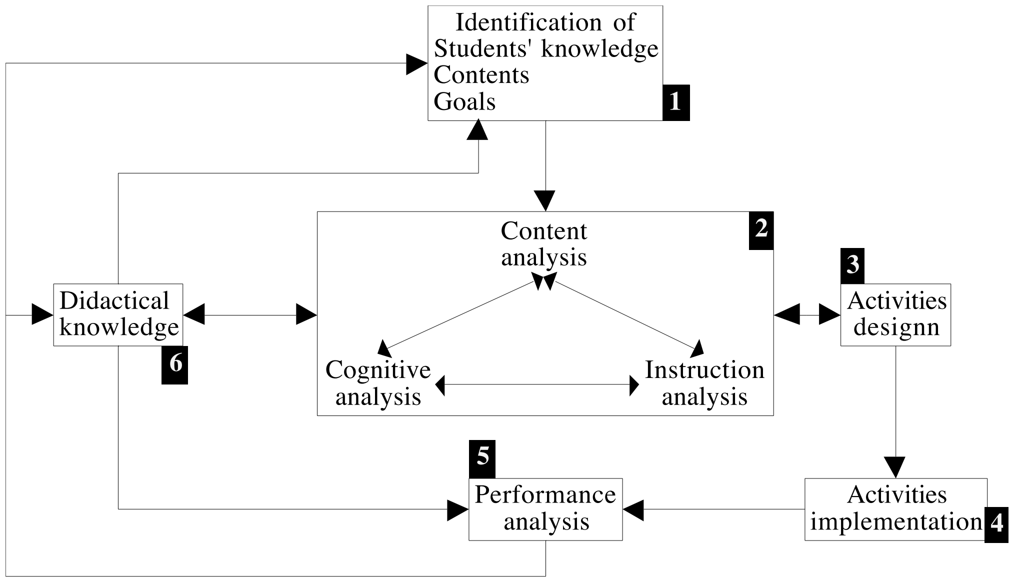

Figure 1. Diagram of a didactical analysis cycle

The next step of the cycle involves the description of the mathematical content from the viewpoint of its teaching and learning in school (box 2). The content analysis stresses the relationship among concepts, highlights its multiple representations, and distinguishes the connections between the elements of the conceptual structure and between those elements and the phenomena from which they emerge. This information is used in the cognitive analysis, in which the teacher describes hypotheses about how students construct their knowledge when they face the learning activities that are proposed to them. The cognitive analysis involves the identification of the skills, reasoning, and strategies (competencies) necessary to solve the tasks, of the mistakes students can make when they are solving them, and of the difficulties and obstacles they might face. The information from the content and cognitive analysis allows the teacher to carry out an instruction analysis: the identification and description of the tasks that can be used in the design of the teaching and learning activities that will compose the instruction in class (box 3). Based on her assumptions about students' learning, the teacher should select those tasks that are coherent with them. For instance, if she follows a cognitive constructivist theory of learning, she might choose those tasks that could mobilize students' knowledge in order to generate cognitive conflicts and promote the construction of meaning using the materials and resources available. In the performance analysis the teacher observes, describes, and analyzes students' performance in order to produce better descriptions of their current knowledge and review the planning in order to start a new cycle (box 5).

We do not have space here to describe in detail each of these analyses. As an example, we describe briefly the first one. Content analysis is the analysis of school mathematics, that is, the mathematics viewed from its school teaching and learning perspective. Content analysis tries to understand the complexity of mathematical subject matter by focusing on its different meanings. We take into account three approaches for this purpose: conceptual structure, representation systems and phenomenological analysis. The conceptual structure is the description, in terms of concepts, procedures and the relationships among them, of the mathematical structure being analyzed (Hiebert & Lefevre, 1986). We see the representation systems as a means for expressing and highlighting different facets of the same mathematical structure and we work with them under the assumption that they follow a sequence of rules originating in mathematics, in general, and in the specific mathematical structure, in particular (Rico, L., Castro, E., & Romero, I., 1996). The phenomenological analysis involves the identification of the phenomena that are in the base of the concepts, the situations that can be modelled by the mathematical structure, the substructures of that structure that serve as models for those phenomena and situations, and the relationships between substructures and phenomena (Freudenthal, 1983).

Didactical knowledge

The didactical knowledge is the knowledge that the teacher enacts when she performs the didactical analysis (box 6). In other words, it is the knowledge needed for organizing teaching and learning activities. Expert teachers perform didactical analysis based on their experience and the materials they have available. However, pre-service teachers need guidelines and criteria with which to organize their activities, produce their work, and structure their future experience. The didactical analysis provides guidelines for pre-service teachers to 1) explore and recognize the richness and variety of meanings of the mathematics subject matter, 2) collect, organize and select information concerning these multiple meanings, and 3) use this information to design materials and activities to promote students' mathematical learning. The knowledge underlying these guidelines can be organized in three categories: 1) the concept of curriculum as a tool for planning and global structuring, 2) the foundations of school mathematics (mathematics, learning, teaching, and assessment), and 3) mathematics education notions that can be used, as conceptual and methodological tools, for local planning, as suggested in the cycle of didactical analysis. We call these notions "curriculum organizers" (Rico et al., 1997). Didactical knowledge is the integration of these three types of knowledge.

The notion of curriculum serves as the global structuring idea for the overall planning process. On the other hand, in order to be able to justify and establish a proper framework for the curricular design, the teacher has to be aware of the different standpoints concerning mathematics in school and its teaching and learning and she has to assume explicitly one of them. In this document we will focus on the curriculum organizers as the conceptual and methodological tools that, once a stance has been taken concerning mathematics, its teaching an learning, allow the teacher to provide specific meaning to the curriculum design of a particular mathematical subject matter. As notions, the curriculum organizers emerge from the knowledge that mathematics education has developed as a discipline. As tools, they provide the teacher with a conceptual and methodological apparatus for analyzing the mathematics subject matter at hand, revealing it's conceptual, cognitive, instructional and performance meanings, and organizing that information in order to produce a systematic and justifiable lesson planning. Therefore, we expect the teacher to develop both a theoretical and a practical knowledge: theoretical, in the sense of knowing the conceptual aspects of each notion according to what has been established in the mathematics education literature; and practical, in the sense of her ability to put those notions into play for analyzing a specific mathematical subject matter.

We have chosen a set of mathematics education notions as curriculum organizers for each of the analyses in the didactical analysis cycle. We have shown above that, for the content analysis, we propose three notions: conceptual structure, representation systems and phenomenological analysis. Similarly, in the cognitive analysis we focus the attention on the students' competencies and on the notions of errors and difficulties, while the instructional analysis takes into account the materials and resources available and the notion of modelling for problem solving. The performance analysis uses the notion of assessment and takes into account the information produced in the cognitive analysis. Finally, the history of the mathematical subject matter is an underlying notion that provides information for all the analyses. We do not have space here to describe the curriculum organizers in detail. Such a description should include our expectations about how the teacher should use each notion within the didactical analysis process, and how she should articulate the information produced for lesson planning purposes.

Didactical and mathematical knowledge in pre-service training

The curriculum organizers are mathematics education notions and therefore they cannot be considered as mathematical knowledge, at least from the viewpoint of a large proportion of mathematicians. Nevertheless, putting them into practice for performing the didactical analysis requires more than the conceptual knowledge that emerges from their mathematics education meaning. In order to analyze a specific mathematical subject, the teacher needs to be able to examine and enquire about the mathematical structure supporting the subject matter at hand. In the case of the content analysis, this seems to be clear. Analyzing the mathematical structure in terms of the concepts, procedures and relationships involved, requires a deep knowledge of the mathematics behind that structure. Similarly, a good mathematical knowledge is needed for identifying the representation systems that can be used for representing the mathematical structure, the relationships among them, as well the phenomena related to that structure and the substructures of that structure that can be used to model those phenomena.

Cognitive analysis requires a general knowledge about learning theories and a particular knowledge about the specific cognitive issues related to the teaching and learning of the subject matter in school. Teaching experience usually provides some information on this matter. The teacher has to complete this information with the material found in the mathematics education literature. Interpreting, analyzing and organizing this information for lesson planning purposes require mathematical knowledge. The information that the teacher collects for the cognitive analysis is specific to the mathematical structure at hand. Interpreting it for teaching purposes requires its integration to the information concerning the mathematical structure itself. In other words, the teacher has to put into play her mathematical knowledge in order to produce a useful and meaningful cognitive analysis. A similar argument can be put forward concerning the instructional and performance analyses. Once the teacher gets into the specificity of a particular subject matter, the didactical analysis puts permanently into play her mathematical knowledge. On the other hand, the teacher can articulate the information produced while performing the didactical analysis with her mathematical knowledge. That information can be seen as knowledge about a mathematical structure, and it allows the teacher to structure its multiple meanings from the viewpoint of its teaching and learning.

We have proposed didactical analysis as a procedure that the teacher can use for identifying and organizing the multiple meanings of a mathematical structure in school mathematics in order to produce a lesson planning. We have suggested that, when working on a specific mathematical structure, these didactical meanings are in close connection to the teacher's mathematical knowledge. The more she integrates these two types of knowledge, the better the lesson planning she can produce. That is why, one should not split the teacher's mathematical knowledge and her didactical knowledge into two independent domains. In fact, mathematician's and teacher's training should try to integrate both types of knowledge. This is not usually the case.

In Spain, mathematicians' training is usually focused on a formal and symbolic approach to mathematics. Students rarely get in touch with non-symbolic representation systems, neither get involved into historical, epistemological or phenomenological analysis of the subject matter. Cognitive or instructional discussions are even scarcer in their mathematics training. When faced with the didactical analysis of a mathematical structure in their pre-service training at the end of their mathematics studies, they usually revert to their secondary mathematics textbooks. On the other hand, by identifying and organizing the multiple meanings of a mathematical structure, they finally recognize the complexity involved in a subject matter they initially considered as extremely simple. In this sense, they improve their mathematical knowledge, even though this development is not always made explicit in their pre-service training.

THE COURSE IN PRACTICE

The ideal model we have described supports a pre-service teachers' training methodology course whose aim is to develop trainee's competencies on secondary mathematics curriculum design. The content of the course is the didactical knowledge as described in this paper. Pre-service teachers work in groups. Each group puts into play the didactical knowledge notions on a chosen mathematical topic, presenting the results of their work regularly to the whole class. At the end of the course, each group produces a curriculum design that is expected to be justified on the information they collected and analysed for each curriculum organiser.

Preliminary results from an ongoing study show an evolution of pre-service teachers' didactical knowledge over time. We have characterised this evolution on the basis of four stages (Gómez & Rico, 2004). The results show that pre-service teachers are able to manage the notions of didactical knowledge as tools for collecting information on and organising the different meanings of a mathematical concept. However, some of them have difficulties in using this information in the design of learning activities. Nevertheless, pre-service teachers recognise the complexity involved in the teaching and learning mathematics in school and develop competencies for analysing and criticising curriculum design proposals.

Pre-service teachers come to the methodology course with different learning histories, attitudes and expectations. We have already mentioned the implications of their "traditional" mathematics training. Since they have very little teaching experience (if none at all) they are mainly concerned with class management issues and expect ready to use recipes for teaching. They have therefore difficulties grasping the idea that a set of theoretical notions (the curriculum organisers) can have useful practical implications. This difficulty concerning the duality between theoretical and practical knowledge is enhanced by the fact that the course itself is mainly concerned with curriculum design, and does not deal with the practical aspects of class management.

As it is usual in this type of course, we have to adapt its contents and methodology to the time available. We have also to plan pre-service teachers activities to the availability of Spanish mathematics education literature.

Discussion

We have given a partial answer to the question “What mathematics should teachers in training study?” for the case of pre-service secondary mathematics teachers. Even though they usually have better mathematics training than primary teachers do, we have suggested that such a knowledge is not sufficient. For that purpose, we have identified a set of activities that we expect teachers to perform in order to produce lesson planning. From those activities, we have characterized the didactical knowledge that we consider relevant in pre-service secondary mathematics teacher training. We have indicated that there should be a closer relationship between this didactical knowledge and the teachers’ mathematical knowledge, proposing, therefore, a partial answer to another issue: “How best can we integrate content knowledge and didactical knowledge for teachers?”

From a teacher perspective, knowing mathematics does not only mean knowing the mathematics of a mathematics' major or the school mathematics as they are traditionally proposed in school textbooks. Knowing mathematics for teaching means as well being able to analyse a specific subject matter in order to reveal its multiple content, cognitive, instructional and performance meanings. In this sense, didactical knowledge and content knowledge cannot be separated. They can be seen as the knowledge required to approach school mathematics from an “enlarged pluralistic” perspective.

It is not clear whether mathematics teachers training programs should focus on trying to affect teachers’ beliefs as a means for improving teaching (Lerman, 2001). Whatever her beliefs, the teacher should make them explicit and be coherent with them when planning and implementing lessons. Our experience as teachers’ trainers has shown that teachers become more conscious of their beliefs and recognise conflicts between those beliefs and their own experience, when they are able to approach school mathematics with conceptual and methodological tools as the curriculum organisers presented here.

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