The role of representation systems in the learning of numerical structures

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1. Introduction

The learning of number concepts within the school system and the role of the representation notion has to analyze and interpret the understanding of number concepts in schoolchildren are important topics in numerical thinking research. Our research team is interested in the difficulties young people find on managing numerical structures when they face advanced mathematical questions. The work presented here will show the general aims and some results of a piece of research done by our team in this field.

We have chosen the representation concept to point out some curricular lacks and to observe students' work on learning numerical concepts, and consequently to interpret their numerical thinking construction. Such an idea has been continuously considered as it is interesting and useful for the mathematics education researchers (Janvier, 1978, 1987; Kaput, 1987, 1992; Goldin, 1993; Duval 1993, 1995). Though most of the considerations we make here are suitable for other mathematical subjects, our actual contribution is limited to numerical concepts and structures.

Since there is no an univocal meaning for this term, it is important to specify the sense in which we are going to use the representation concept. We will carry out such task discussing three different approaches to this concept. Once we have done this, we will present the results of our research.

2. About the representation notion

2.1. General features

The history of both philosophy and science show the richness of the different interpretations that this concept has (Ferrater, 1981). Some of them are interesting for current lines of research in mathematics education.

A first point of interest for us is to underline that the representation idea implies something to be represented. It is generally assumed that *any concept of representation must involve two related but functionally separated entities*. One of these entities is called the *representing world* or *representation* and the other is the *represented world,* which implicitly presupposes some kind of connection between the objects of the representing world and the objects of the represented world.

Thus "*any particular specification of a representation should describe the following five entities:*

1º the represented world
2º the representing world
3º what aspects of the represented world are being represented,

4º what aspects of the representing world are doing the representing
5º the correspondence between the two worlds
In many of the interesting cases one or both of the worlds can be hypothetical entities or even abstractions" (Kaput, 1987).

Therefore we consider necessary to distinguish the representation systems from the numerical concepts and structures for which they stand. When we identify natural numbers with the numerals that we get by the writing rules of numbers of the decimal system, we forget that the decimal system is only a way of writing numbers, statements and proofs by lineal combination of successive powers of 10. Using the Arithmetic's Fundamental Theorem, it is possible to write each number as a product of prime factors, and this shows its multiplicative structure; this is another representation system for natural numbers.

Though it is not usual, we will consider which different features and properties of natural numbers are highlighted by each kind of symbolization. Each of the natural numbers representation, together with its own rules, proposes a different description of the natural number concept. It is a simplification to identify numbers with any of its notations and what is worse it is inadequate for mathematics education research. So, we will differentiate between numbers and its kinds of representation.

A second important idea is the contemporary philosophical use of the representation term to refer to *anything that can be semantically evaluated* (Dancing & Sosa, 1993). It can be said that representations are true, that they refer to, that they are true with regard to something, that they are about something, that they are accurate and so on. *Contents* is the technical term used for naming *what makes a representation semantically evaluable*; thus of a statement it is said that sometimes it has a proposition or truth condition as its content; of a term it is said that it has a concept as its content; of a graphic, that it expresses a proper relationship between its elements. A representation's content is just whatever it is that underwrites its semantic evaluation. From this point of view, symbol expressions, statements, diagrams, graphics, tables and other common notations are mathematical representations.

2.2 Numerical structures and representation systems

Current number conceptualization is based on the system notion; talking accurately we are not just referring to number concepts but to number systems or structures. A numerical structure is a set of abstract entities expressed symbolically, provided with operations or ways of composing numbers and with relationships to make the comparison among its entities possible. What characterizes a numerical structure is the consideration as a whole of its entities, their operations and their relationships (Feferman, 1989). For number systems a rather small collection of big and powerful ideas determine the structure of each system (Fey, 1990).

Mathematicians work with meaningful symbols and representations (Kaput, 1987) whose nature and use has been of great interest for mathematics thinkers and researchers along the history of this discipline.

The set of signs, symbols and rules to express or represent a numerical structure must satisfy its systemic nature. That is why we can hear about *sign mathematical systems* (Kieran & Filloy, 1989), *notation systems* (Kaput, 1992) or *semiotic systems* (Duval, 1993). We prefer to use the term *representation systems* when talking about the several modes of expressing and symbolizing numerical structures by means of some specific signs, rules and statements. The decimal numeration system is a paradigmatic example of a well known representation system for natural numbers.

The structural consideration of numbers and our choice of distinguishing between numbers and their representations lead us to the formalist foundation of mathematics. Numerical fields are established as operative fields by the formal approach of Peano and Hilbert (Badiou, 1990). The formalist foundation of mathematics stresses a technical consideration of numbers, as some kind of tools to carry out some processes, following some rules and with the possibility of establishing a variety of relations among numbers. In the formalist school, signs and symbols play a central role, together with the syntactic rules by means of which they combine to cause more complex expressions and formulas, which are necessarily complemented by finitist procedures to prove statements and formulas of each numerical system (Von Neumann, 1964).

On the other hand, in our position about its epistemological base, mathematical concepts do not refer to objects or physical phenomena but to the relations among objects, phenomena or concepts, and consider mathematical concepts as abstract entities that need to be expressed by some symbolic system; that is to say, mathematical concepts are given by means of one or several specific representations. We consider two different levels of representation: facts or particular concepts (i.e., the unity) represented by specific symbols (i.e., 1), and the relationships between concepts (i.e., one plus one makes two) represented by symbolic statements (i.e., $1 + 1 = 2$) (Körner, 1977). We assume a phenomenological base for the numerical concepts and relations.

Besides, there is not a symbolic system completely suitable to express the complexity contained in each mathematical concept; this is the reason why each concept has more than one representational system which at the same time emphasizes and sets out some important properties but also blurs or makes other properties more difficult to understand. We accept as mathematical representation systems: natural language, drawings and graphics, different symbolical writings, tables and the algorithmic notations which describe an operating rule.

2.3 Representation and cognition

In mathematics education mathematical concepts should be linked with the mental activity of human beings. Following Wittgenstein when he analyzes several mathematical language games and among them the number concept (Wittgenstein, 1988; §§ 65-68), we claim that every mathematical concept is supported by its different uses and meanings and so by its representations. All this in the sense that the use of each concept is what establishes its semantic field by extension and that each other meaningful mode of understanding a concept needs its own symbolization system or representation to be recognized. This leads us to the well-known distinction between external and internal representations. Internal representations or thinking objects, which are supposed to be placed within individual human minds, are different from external representations whose semiotic character is given by signs, symbols or graphics.

The wide use of the representation notion to characterize human mental conditions and activities is an outstanding feature in the current development of Cognitive Psychology (Guttenplan; 1994). We assume that cognitive processes are those that deal with representations. What establishes the difference between cognitive processes and those that are not is exactly that the former but not the latter can be epistemically evaluated. Since only something with contents can be epistemically evaluated, only processes can be considered as being cognitive as they involve representations. A proper internal domain of external representations is essential in the development of numerical thinking processes; this is a basic tenet for the understanding of number concepts in human beings.

We consider understanding as a representation, which is structurally or conceptually directed, of the relationships between the pieces of information that should be learnt, and between that information and those ideas and our knowledge and experience basis (Wittrock, 1990). We admit that different subjects present different understanding about the same concept or mathematical structure because their representations have different contents. The links between external and internal representation are clues to study understanding phenomena.

2.4 Balance

The representation concept in mathematics education must consider its duality. "*To think about and to communicate mathematical ideas we need to represent them in some way. Communication requires that the representations be external, taking the form of spoken language, written symbols, pictures or physical objects. (...) To think about mathematical ideas we need to represent them internally; in a way that allows the mind to operate on them*" (Hiebert & Carpenter, 1992).

Mathematical knowledge is only reachable by external representations, which are the facts for this knowledge. Representation is also involved in the actual working of our thought, and it has a central position in the learning of mathematics.

This duality of the concept converts it in a suitable tool to study understanding phenomena; for the researchers' aims it is useful when deciding to inquiry on the different ways by which human beings process numerical structure.

We have decided to use the term *representation systems* though we are aware of the problems which have been pointed out by Kaput (1992). He considers this term leads to the distinction between the representation system (representing) and the numerical concept (represented), and so it is necessary a self definition for the second one. Nevertheless we consider that we have the same problem if we talk about signs and symbols instead of representations, because symbols must express or denote a concept whose characterization has to be done outside these notations, at least from a non-nominalistic point of view. This is why we have discussed some of the previous ideas.

From the analyzed complexity we have been able to emphasize phenomenological and cognitive dimensions of numerical thinking. We have also been able to move away from the platonic foundation which claims for the reality of mathematical concepts out of space and temporary conditions and also out of human beings mental activity (Kitcher, 1984).

3. Scope of this work

3.1 Background

At the beginning of 80's there are two conceptual fields whose study is based on the notion of representation.

One of these fields is related to the concept of function; the studies that have been carried out emphasize the different systems for functions representation and detect some difficulties for the understanding of this concept due to translation problems among these systems. Among the most famous are Janvier's works, which ended in his thesis in 1978, and which later were used for the materials created by the Shell Center in Nottigham University. These materials undertake a kind of diagnostic teaching on this field, based on graphic representations.

The second of these fields deals with the concept of rational number, considering and analyzing different representation systems for this number field. Behr, Lesh, Post and Silver's works (1983) are among the pioneer ones in the study of this number set, which is still offering useful results.

In 1984 a symposium is held in the University of Quebec (Montreal) organized by CIRADE, in order to present and discuss the last stages of a research project on representation. The result of this symposium is the book "*Problems of Representation in the Teaching and Learning of Mathematics*" (1987), where it is discussed the usefulness of the concept of representation in mathematics education .

The interest in the topic is specially shown by the existence in the *International Group for the Psychology of Mathematics Education* till 1995 of the *Working Group on Representations*. Goldin, who was the coordinator for this working group, expresses the general concern on this topic: "*Representations are a key theoretical construction in the psychology of mathematics education. The meaning of this term is quite broad and it includes:*

a) external physical embodiments (including computer environments): an external structured physical situation or set of situations that can be described mathematically or seen as embodying a mathematical idea;

b) linguistic embodiments: verbal, syntactic and related semantic aspects of the language in which problems are posed and mathematics discussed;

c) formal mathematics constructs: a different meaning of representation, still with emphases on a problem environment external to the individual, is that of a formal structural or mathematical analysis of a situation or sets of situations;

d) internal cognitive representations: very important emphases include students' internal, individual representation(s) for mathematical ideas, such as "area", "functions", etc., as well as systems of cognitive representation in a broader sense that can describe the processes of human learning and problem solving in mathematics" (1993).

From a semiotic approach Duval, from the University of Strasbourg, is working since the beginning of the 80's on the representation notion and on the understanding of mathematical objects; his work *Semiosis and Noesis* (1993) is a valuable contribution in this sense.

Nevertheless, we have not found any previous work on the representation systems for natural numbers neither on the understanding of the general term of a sequence which have been based on these several representation systems.

3.2. Aims and assumptions

The main aim of this work is to make clear the plurality of representation systems by which number structures are expressed. We maintain that each number system, as a complex set of entities, relationships and operations, cannot be expressed as a whole by only one representation system. Conventional number structures need the coordinated action of several representation systems in order to underline essential features of such structures. Particularly, graphic representations play an important role in understanding number structures. These are some of the conclusions in our work "*Exploring number patterns by means of point configurations*" (Castro, 1994). Here we study the integration of three representation system for natural numbers in order to deepen on the concepts and procedures used by 12-14 years old students in relation to the notion of general term of a sequence of natural numbers .

4. Representation Systems for Natural Numbers

4.1 Decimal numeration system

Decimal numeration system is a powerful mathematical tool, the result of a long historical evolution, inspired by economical principles, not only semiotic but also operational, by which men have developed and expressed their counting, classifying, measuring and ordering skills. In our current society the domain of this system is a basic cultural fact; its knowledge establishes one of the criteria to determine that a human being has acquired the basic skills that allow him/her to hold a deserving intellectual position in society. That is why educational systems give such a value to transmitting and learning decimal numeration and basic arithmetic operations, using the decimal numeration system as the only one.

This is the way we come to identify each number with its decimal notation and the set of natural numbers with their Arabic notations. Such an identification, although culturally useful and economical, is still a limitation for the learning of natural numbers.

4.2 Arithmetical analysis

The dynamic character of the natural numerical system gets blocked by the inertia of the common decimal representation; its dynamism requires that numbers be determined by their intertwined relationships. So, knowing, for instance, what 15 means is not just reading it as 1 ten and 5 units, but also interpreting it as 3 times 5, 5 times 3, next to 14, preceding 16, the sum of two consecutive numbers: 7+8, the sum of three consecutive numbers: 4+5+6, the sum of five consecutive numbers: 1+2+3+4+5, coming before a square 4^2 -1, the sum of two numbers multiplied by their difference:$(4+1)\cdot(4-1)$, half of 30 and so on. From this point of view, each number is a knot in which several relationships intertwine, it is an element of a complex net closely connected whose wider or smaller domain will determine the real understanding reached by each subject in the natural numbers system (Rico, 1995).

The former considerations show that, on the basis of decimal notation, there are other representation systems for natural numbers. The *arithmetical analysis* of numbers is one of them; this analysis consists of considering each number as a sum or as a product of simpler numbers. The former examples are expressions of number 15 by means of its arithmetical analysis.

4.3 Graphical systems

Nevertheless, we still have not taken into account ways of graphic representation. The number line is the standard graphic representation. We choose two points arbitrarily on a line and we give them the values 0 and 1; by agreement, the point that matches 0 is on the left of the point that matches 1:

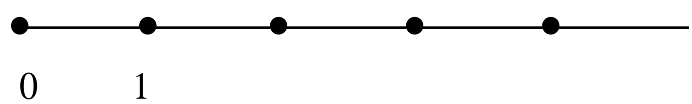

Fig. 1

From 1 to the right we write down points that keep the same distance between them than that of the two initial points and mark on them consecutive natural numbers. This representation is an useful tool to understand numbers, carefully studied for the domain of the natural numbers system (Resnick, 1983).

4.4. Point configurations

History sets us in touch with another powerful system of representation for natural numbers that has been ignored by the current school mathematical curriculum. We are talking about point configurations used to represent figurative numbers and whose origin and development was the Pythagorean number concept.

For those who followed the Pythagorean doctrine a number was not just a label for a collection, the symbol for a quantity or an intellectual construction, but something which was consistent by itself; numbers were like atoms that, in their varied compositions and relationships, gave the essence itself of the plural real world.

This number notion had its best expression in the representation that we know as *point configurations* or *figurative numbers*, completely different from the usual numeration systems.

The basic idea of this representation system is considering each number as an aggregate of points or units distributed on a rectangular or isometric scheme and according to a flat or space geometric figure. This way triangle, square, rectangle, pentagon, pyramid and cubic numbers come across, as many as different geometric figures are considered; they allow us to think in each number as a whole arranged with regard to a fixed geometrical structure

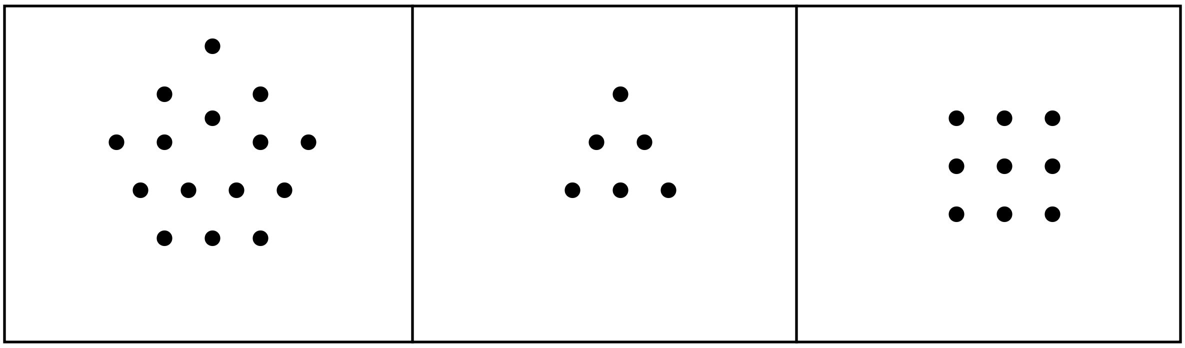

Fig. 2

To summarize:

Point configurations are a number representation system based on:

*a single symbol: the point;

*a structured space of representation, commonly the square or isometric scheme when working on the plane;

* a way of arranging the quantity of points that satisfies some agreed criteria of symmetry or regularity and that can be explicited easily.

These three conditions establish a new representation system for numbers (Beiler, 1966), whose advantage lays in providing graphic models which help to visualize and analyze the arithmetic structures of each number.

Two important pieces of information emerge when arranging, geometrically, the units that embody a number. On the one hand, we see an arithmetical analysis of the number: a triangular number is the sum of consecutive numbers starting from 1, a square number is the product of a number by itself, a rectangular number is the product of two consecutive numbers. This visual analysis allows us to know several properties of each number and relate it with many others. Besides, the same number could be considered as belonging to several kinds of figurative numbers.

On the other hand, different numbers share the structure that represents each kind of point configurations. The shared configuration shows several arithmetical analysis and each one becomes a common property for all these numbers; this property can be generalized. So the representation system of point configurations becomes a useful tool to establish general properties of numbers and find new relationships among them.

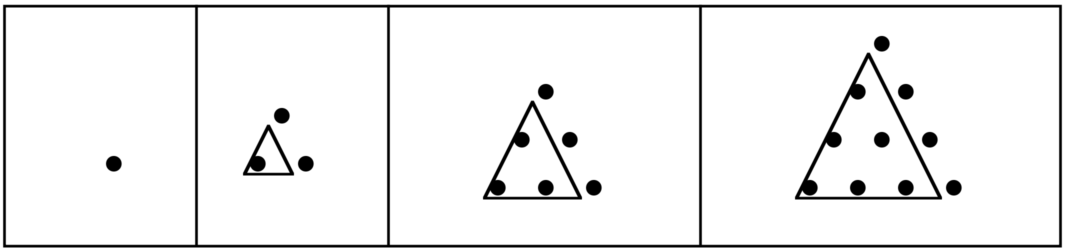

Fig. 3

Historically this representation system allowed mathematicians to establish general number properties and algebraic identities without having the current symbolic signs of algebra. We can find the use of figurative numbers along the Theory of Numbers history.

5. A curricular problem

5.1 Starting with sequences

The concept of sequence of natural numbers is a complex one; it is based on two notions: the one of a totally ordered set and the one of infinite process, by which every term of the sequence has a following one. When we have several terms of the sequence and we are challenged to go on with it, the proposal is to find new numbers related with the ones just known trying to use the same relations they have among them. There are several ways to relate a limited number of terms; to the question: "1, 2, 4, .., which is the next term?" there is a multiplicity of feasible answers (Sloane, 1973). The possibilities of finding new relations among the terms of a sequence are decreasing when its number is increasing, and the options to continue the sequence could be reduced to a single one. Recognizing the relations among the given terms of a sequence can let the students find new ones, that is to say, continue the sequence. Nevertheless, the characterization of a sequence is given by its general term.

5.2 The general term of a sequence

To find and express the general term of a sequence offer some understanding difficulties, and there are many students who are not able to find a proper meaning for this idea because the high abstraction level implied on it. What does the general term of a sequence mean? The general term of a sequence is the algebraic expression of the rule which is followed by all its terms taking into account its corresponding ordinal place. The general term of a sequence expresses the common structure shared by all its terms when they are considered as members of an ordered set. The usual way to write the general term of a sequence is by means of algebraic notation. So, the formula $a_n = (n^2 + 2n)/2$ expresses that all of the terms of the considered sequence can be found taking the square of its ordinal, adding up its double and dividing the result by 2. Nevertheless, this idea of a common structure or *the shared structure of all the members of the sequence* cannot be captured by the analysis of the relations among two or three consecutive terms.

To have several numbers written in the decimal system at your disposal does not allow us to observe the common structure they have; in order to know this structure is necessary to have the numbers written by a shared arithmetical analysis or, better than this, to have them expressed by means of point configurations following a single pattern. Triangular numbers visualization shows that the numbers 1, 3, 6, 10, 15, ...share a common pattern (fig. 2). The arithmetic version of the pattern:

$$1,\ 1 + 2;\ \ 1 + 2 + 3;\ 1 + 2 + 3 + 4;\ 1 + 2 + 3 + 4 + 5;\ ...$$

advances the shared structure by means of the first numbers of the sequence: each one is the result of summing up consecutive numbers from 1 on till the corresponding number to its ordinal position in the sequence. But it is still necessary to consider many understanding phenomena to establish that the general rule of this sequence is, precisely, $a_n = (n^2 + 2n)/2$.

5.3 Curricular context

The work "*Exploring number patterns by means of point configurations*" (Castro, 1994) poses and studies the viability of a representation system for natural numbers, as an adequate tool for visualizing and analyzing sequences, similar to the graphic representation of functions, in the mathematical curriculum of the Compulsory Secondary Education. We study the strength of point configurations to express numerical relations and properties; we also study how students discover and use the numerical properties by means of such representations.

Our study is summarized in the following considerations:

* the coordinated use of three representation systems for natural numbers: point configurations, decimal numeration system and arithmetical analysis or development of numbers;

* the work and reflection on the pattern by which linear and quadratic sequences are defined on terms of point configurations and arithmetical development;

* the performance of the following tasks: to continue a sequence, to extrapolate terms; to generalize; to find out the general term and use it to obtain specific ones.

6. Findings and discussion

6.1 Sequences and representation systems

Point configurations allow us to represent sequences of first and second degree taking integer values by means of a graphic display. So, arithmetic progressions allow simple point configurations, generally with rectangular shape and with constant base or height. They are called linear sequences because the pattern representation of their terms can be analyzed decomposing them by lines, and the difference between two consecutive terms can be described as the aggregation of a line.

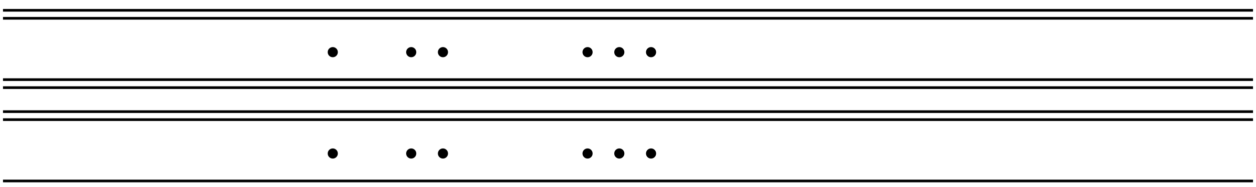

Fig. 4

The sequences with constant second differences are those whose general term is expressed by a second degree polynomial function. The simpler cases are the sequence of square numbers: $C_n = n^2$, and the sequence of rectangular numbers: $R_n = n\,(n + 1)$.

It is possible to make a graphical representation of theses sequences having in mind that its two dimensions vary; the change from one term to the following is defined by its growth in both dimensions. The structure of these numbers is called quadratic and the change from one term to the following is not constant but variable with a linear variation.

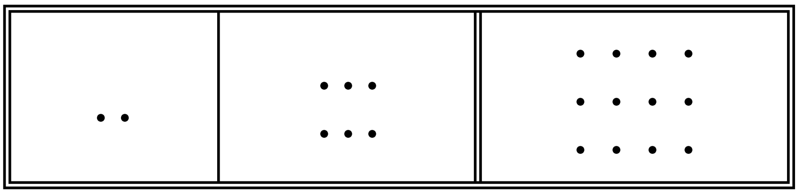

Fig. 5

In general, if the rule of a sequence is $a_n = an^2 + bn + c$, such a sequence has a constant second difference. If a_n takes natural values for every n, then each of its terms can be represented by a polygonal plane configuration, regular o irregular, and all its terms have the same pattern of representation.

6.2 The diversity of analysis

We have introduced 12-14 year old secondary education students to the symbolic representation system of point configurations. We have used these representations as an alternative symbolic system for the purpose to carry out the following tasks: to visualize the representation pattern shared by the terms of the sequence, to continue the sequence and represent some advanced terms with the pattern. Thus, in the example of figure 5, students recognize the geometrical shape shared by the three represented terms, they are able to add the two or three following terms and also to represent the 11th or 15th terms.

Likewise, point representation provides a structural analysis of the terms of the sequence and allows us to express new terms by means of the arithmetic analysis obtained. For the figure 5 example, there are several correct arithmetic analysis found by the students:

a) 2, 3 + 3, 4 + 4 + 4, 5 + 5 + 5 + 5, ..

b) 1 + 1, 2 + 2 + 2, 3 + 3 + 3 + 3, 4 + 4 + 4 + 4 + 4, ...

c) 2 x 1, 3 x 2, 4 x 3, 5 x 4, ...

d) $1^2 + 1$, $2^2 + 2$, $3^2 + 3$, $4^2 + 4$, ...

e) $2^2 - 2$, $3^2 - 3$, $4^2 - 4$, $5^2 - 5$, ...

We can observe that there is a variety of different analysis of the point configuration pattern; and each one provides a possible arithmetic development (sometimes additive and sometimes mutiplicative) which is shared by all the terms of the sequence. In this way it is possible to obtain several expressions for the terms of the considered sequence, with the representation system we have called arithmetic analysis. When the same sequence is displayed in the decimal numeration system: 2, 6, 12, 20,.... students have not enough information to find a common arithmetic development for all these terms.

6.3 Findings

We have studied how the students understand and generalize the common structure that the terms of a sequence have using the established connections among the terms of the point configuration sequence and the terms of its arithmetic development. That is to say, we have tried to explicit the general term of a sequence notion that 12-14 year old students have by means of the question "How can we write the n-th term?" The answers to this question are different according to the representation system considered.

So, in the decimal number system, the most common expression given for the general term is **n**, which is the immediate symbolic translation for the expressions: "a number in general", "any number of the sequence", "any term of the sequence", or the like.

When the geometrical pattern is used, as it is necessary to leave some wide spaces between the points to indicate the generalization to **n**, this leads some students to change the model by a continuous shape for the general term. That indicates the difficulty of this representation system for expressing the general term.

In representations by means of the arithmetic development system we find that it is not difficult to move to the general term in a successful way. Nevertheless, when pupils have several arithmetic expressions about the terms of the same sequence it is not easy to accept as equivalents the general term expressions obtained.

A strong obstacle for finding the expression of the general term of a sequence has been detected in this study. Both, the point representations and the arithmetic developments, express in some way the structure shared by several numbers. The decimal notation of the same numbers does not allow us to capture the common structure.

When we ask for obtaining the general term of a sequence we really ask for the general expression of the common structure of all its terms, by means of an algebraic symbolism. Because in the number decimal system each term is shown by a single symbol and the common structure is not considered, the former question (how can we write the n-th term?) cannot be answered in this system. This explains that the most common given answer is "**n**", which is a single symbol and expresses "a general term". With the point configurations representation system it is possible to appreciate the common structure, but the concrete manner of such representations makes the finding of a general term difficult. Only with the arithmetic analysis system it is possible to generalize the expressions of a sequence terms.

The question: "Which is the general term of this sequence?" is a more abstract version of the question "How can we write the n-th term?" and it has its answer on the arithmetic development representation system and it has no answer in decimal system. There are very few students in our study capable of understanding the question, because many times it is posed in the former system (decimal) and it must be replied necessarily in the latter one (arithmetic analysis).

Only with the integration of the representation systems, as different expressions of the same idea, it is possible to talk about the understanding of the concepts of a sequence and its general term.

7 Conclusion

We have introduced 12-14 year old secondary school students to the point configurations representation system. For that purpose we have used these representations as an alternative symbolic system to carry out a structural analysis of numbers sharing the same visual representation pattern; in this way we have obtained the arithmetic development shared by the terms of the same sequence.

Our work focused on the study of linear and quadratic natural numbers sequence by using the three symbolic systems just mentioned: figurative, decimal number and operational or arithmetic development. Thus, it is possible to stress the development patterns of point sequences as well as number sequences. We have considered point configurations as models that express development patterns of number sequences, showing the lack of representation of these contents in the decimal system writing.

The results obtained in our study have highlighted that students accept, without difficulty, the point configurations system for numbers and they use it properly working with different geometrical models; students find a great variety of relations for triangular and squared numbers and they establish arguments to connect the geometrical pattern with its respective arithmetic translations by means of point configurations.

The data provided by the students, from the proposed tasks, have shown that the most intuitive of the three representation systems is the point configurations one due to its graphic character, which favor a visual analysis and allows the processing of the quantity structure. However the maximum strength of this system is reached when it is conjointly used with the arithmetic developments and the current decimal number system. A point configuration is meaningful when it is used as the visualization of a singular arithmetic development for a specific number (or a family of numbers). The variety of developments suggested for the same point configuration shows the intuitive character of this system.

To the arithmetic level, the new symbolic system provides an operative character to natural numbers which is seen while carrying out the assigned tasks; this way a variety of arithmetic developments are performed for every number. The idea that there are numbers with the same *arithmetic structure* is also strengthened; this structure is visualized by a *geometric pattern* and it is expressed by an arithmetical development. This notion is a first step to the generalization of an arithmetic base.

A third aspect related to the students knowledge clearly shown in the data analyzed is the richness of relations performed with numbers which share the same pattern.

We have checked that there is a weak integration among the three symbolic systems and which is specially clear in the low performance obtained with the tasks designed to express the notion of the general term of a sequence. There are very few students able to identify the general term of a sequence with the operative structure shared by the specific terms given for this sequence; this structure is understood more easily when it is expressed by its arithmetic development.

Though students are able to perform a variety of tasks with the new representation systems, we can say that the 12-14 year old students' understanding on the general term notion is virtually nonexistent because we have not appreciated strong connections among the three representation systems neither any kind of structuration among the mental representations corresponding to the different representation systems used. The arguments given by students show that decimal system and arithmetic system are not clearly seen as two views of the same facts, and there are scarcely some translation rules between them. Only a few students, who coordinate more or less the three systems and incorporate them, show a kind of control for the notion though several understanding levels are appreciated.

There is enough evidence to maintain our main hypothesis: the richness of the numerical structures and their complexity need several complementary systems to be understood; the contribution of graphics representations is essential to understand certain structural notions and to develop numerical thinking.

The integration of several representation systems have been necessary to show the difficulties of some concepts as it is the case of the general term of a sequence, and also to establish ways to overcome these difficulties through the understanding of the underlying structures.

Numerical thinking does not end with the study of the different representation systems, which are useful for the development of a concept, though this analysis is an unavoidable step as well as the notion of representation system. The understanding of numerical structures has a complexity which is still unknown and it needs to be carefully explored. This has been the aim of this study.

References

Badiou, A. (1990) *Le Nombre et les nombres*. París: Editions du Seuil.

Behr, M.; Lesh, R.; Post & Silver, E. (1983). Rational Number Concepts, Lesh, R. y Landau, M. (edts.) *Acquisition of Mathematics concepts and processes*. New York: Academic Press.

Beiler, A. (1966). *Recreations in the theory of numbers.* New York: Dover.

Carpenter, T.; Fennema, E. & Romberg, T. (1993). *Rational Numbers. An Integration of Research*. Hillsdale: Lawrence Erlbaum Associates.

Castro, E. (1994). *Exploración de Patrones Numéricos mediante Configuraciones Puntuales. Estudio con escolares de Primer Ciclo de secundaria (12-14 años).* Granada: Doctoral dissertation. University of Granada.

Dancing, J. & Sosa, E. (1993) *A Companion to Epistemology*. Oxford: Basil Blakcwell.

Duval, R. (1993). Semiosis et Noesis, *Lecturas en Didáctica de la Matemática: Escuela Francesa*. México: Sección de Matemática Educativa del CINVESTAV-IPN.

Duval, R. (1995) *Semiosis et Pensée humaine*. Bern: Peter Lang SA.

Feferman, S. (1989) *The Number Systems. Foundations of Algebra and Analysis*. New York: Chelsea Publishing Company.

Fey, J. (1990) Quantity, en Steen, L. (edt.) *On the shoulders of giants. New approaches to numeracy*. Washington: National Academy Press.

Ferrater, J. (1981) *Diccionario de filosofía*. Madrid: Alianza Editorial.

Golding, G. (1993). The IGPME Working Group on Representations. *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education*. Tsukuba: University of Tsukuva.

González, J.L. (1995). *El campo conceptual de los números naturales relativos.* Granada: Doctoral dissertation. University of Granada.
Guttenplan, S. (1994). *A Companion to the Philosophy of Mind* . Oxford: Blackwell.

Hiebert, J. & Carpenter, T. (1992). *Learning and teaching with understanding.* En *Handbook of Research on Mathematics Teaching and Learning*. Grouws, D.A. (Ed.). New York: MacMillan Publishing Company.

Janvier, C. (1978) *The interpretation of complex cartesian graph representing situations- studies and teaching experiments*. Doctoral dissertation.University of Nottingham.
Janvier, C. (Ed.) (1987). *Problems of representations in the teaching and learning of mathematics.* London: Lawrence Earlbaum Associated, publishers.

Kaput, J. (1987). Representation systems and Mathematics. *Problems of representation in the teaching and learning of mathematics*. Janvier, C. (ed). Hillsdale: LEA.
Kaput, J. (1992) Technology and Mathematics Education. En *Handbook of Research on Mathematics Teaching and Learning*. Grouws, D.A. (Ed.). New York: MacMillan Publishing Company.
Kieran, C. & Filloy, E. (1989) El aprendizaje del álgebra escolar desde una perspectiva psicológica.*Enseñanza de las Ciencias Vol. 7 nº 3.*
Kitcher, P. (1984) *The Nature of mathematical knowledge.* Oxford: Oxford University Press
Körner, S. (1977) *Introducción a la filosofía matemática.* México: Siglo XXI.

Resnick, L. (1983). A developmental theory of number understanding. *The development of mathematical thinking*. Ginsburg, H. P. (ed). Orlando: Academid Press.
Rico, L. (1995) *Conocimiento Numérico y Formación del Profesorado*. Granada: Universidad de Granada
Romero, I. (1995). *La introducción del Número Real en Educación Secundaria.* Granada: Doctoral dissertation.University of Granada.

Shell Centre (1986). *The language of functions and graphs: An examination module for secondary school*. Manchester: Joint Matriculation Board.
Sloane, N. (1973) *A handbook of integer sequences*. San Diego CA: Academic Press.

Von Neumann, J. (1983) The Formalist Foundation of Mathematics, en Benacerraf, P. y Putnam, H. (edts.) *Philosophy of mathematics. Selected readings*. Cambridge: Cambridge University Press

Wiggenstein, L. (1988) *Investigaciones filosóficas.* Barcelona: Critica
Wittrock, M. (1990). *La investigación de la enseñanza, III. Profesores y alumnos.* Madrid: Paidós Educador.