CONDITIONAL PROPOSITIONS: PROBLEMATIC PERFORMANCES AND DIDACTIC STRATEGIES

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We share some results obtained from a teaching experiment, where the role of the use of dynamic geometry is important, designed to help students overcome some problematic performances related to the use and comprehension of conditional propositions aspect that affects learning to prove. We describe four strategies designed to aid students to deal with the problematic performances, the instrument used to determine efficiency of the didactic strategies and report our findings.

INTRODUCTION

Future mathematics teachers should understand the structure and use an if-then statement (conditional) correctly because deduction is an important part of the mathematics that should be taught in schools. Being able to decide whether a conditional is true depends not only on content knowledge but also on its logical structure. Using a conditional to verify validity requires comprehending why it leads to something true. An analysis of the problematic performances our pre-service teachers have when learning to prove in a second semester plane geometry course, led us to determine that one of the causes is found precisely in the interpretation or use of conditionals.

We carried out a teaching experiment, in a course with 18 students, intended to help them deal with the use and comprehension of conditionals. Three characteristics distinguished the course: the axiomatic system used was constructed by the group as an inquiry community of practice (Forman, 1996); the theoretic elements of the system rose from the conjectures students formulated as solutions to designed situations; a dynamic geometry program was used as a mediation tool for the process of learning to prove within that axiomatic system (Laborde, 2001; Mariotti, 2001).

SOME THEORETICAL REFERENCES

Interpreting and using conditionals

There exists a tendency to use expressions of the form “if p, q” to mean “if p, q and if not p, not q”; that is, the original statement is interpreted as a biconditional and not merely as a conditional. This habit permeates the mathematics classroom (O´Brien & Overton, 1980; and Knifong, 1974, cited in Laudien, 1999; Hoyles & Küchemann, 2002). Laudien (1999) claims that the problematic performance related to the use and comprehension of conditionals can be the result of students’ classroom experience and their understanding of common everyday sentences that is not consistent with the

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mathematical use. She identified two types of illogical reasoning schemes frequently used by students, named "negation of the antecedent" and "affirmation of the consequent". In the first case, having a conditional and the negation of its antecedent as true statements, students conclude that the negation of the consequent is true; in the second one, if both a conditional and its consequent are considered as true, students believe that the antecedent of the conditional must be true.

Another possible cause for students' problematic performances with respect to conditionals can be the fact that they are rarely asked to determine the validity of a conditional statement with a false antecedent (Hoyles & Küchemann, 2002). This leads them to identify a conditional ($p \rightarrow q$) as a cause-effect relation (if p, q), assigning to it a temporality character believing that p must precede q in time. Deloustal-Jorrand (2002) refers to the case where a conditional expresses a dependency relation between properties of a figure or properties of a mathematical relation as a "causal conception of an implication", which corresponds to statements usually formulated by students to express results they have discovered when exploring a situation with dynamic geometry, in tasks that are limited to finding invariants of a construction.

Problematic performances

The following problematic performances, related to student's competency to prove, were identified throughout a two-year lapse during which one of the researchers taught the plane geometry course.[1]

PP1: Confusion about the relationship between a conditional and the associated conditional statements. An incorrect meaning is assigned to the dependency or inclusion relationship between properties expressed by each proposition of the conditional. The latter is considered as a whole in which there are two unrelated propositions. This leads to treating them or their negations as if their position in a conditional can be exchanged without affecting the corresponding truth value. Therefore, students use the reciprocal as if it were equivalent to the conditional statement, or the inverse as equivalent to the negation of the conditional.

PP2: Incorrect formulation of the antecedent and consequent of a conditional statement. A conditional that does not include all the significant and relevant information to express a property or relation is stated. This situation has risen in two contexts: (a) when expressing a given conditional statement in the if-then form and (b) when enunciating conjectures obtained from empirical explorations with dynamic geometry. In the first case, the problematic performance occurs when students do not mention a condition that is included in the antecedent even if it is explicit in the

[1] A problematic performance identified but not taken into account for this study, due to scarce occurrence in a geometry course, is when conditional is conceived in a restricted way, considering that it only refers to cases when the antecedent is true.

initial formulation (PP2a); in the second, when the conjecture is inconsistent with the construction process because the antecedent has the properties that arise from the construction, not those constructed, or it does not include all the conditions implicitly established in the construction process (PP2b).

PP3: Expression of a conjecture obtained with the use of dynamic geometry that does not correspond to the process performed. The dependency relation expressed in a conditional is incorrect or incomplete. This happens in two moments: when the conjecture formulated by the students establishes a general fact induced from a particular case evidenced in the exploration (PP3a), and when the conjecture does not report all the conditions that can be inferred from the construction (PP3b).

PP4: Use of a postulate, theorem or definition without having the conditions the respective hypothesis requires. In a deductive step, all the conditions expressed in the antecedent of a conditional are not accounted for. This leads to the incorrect use of the reasoning scheme modus ponendo ponens (MPP).

MAIN CHARACTERISTICS OF THE TEACHING EXPERIMENT

Dynamic geometry

During the teaching experiment, we used dynamic geometry to help students: (i) understand the logical development of a proof; (ii) develop ideas for a proof; (iii) understand dependency relations between properties; (iv) determine the truthfulness of conjectures formulated by others; (v) discover geometric relations between the parts that conform a figure; (vi) impulse creativity in determining auxiliary constructions which are useful for the proof of a theorem. Using dynamic geometry favors comprehensive learning because it requires mathematical knowledge and makes the student's internal context visible, a necessary element in a social knowledge construction process. Since the software (Cabri) installed in the calculators embeds knowledge and gives instantaneous mathematical feedback (Mariotti, 1997), it is in itself what Brousseau calls a milieu because it is a material element on which students can act and which can modify their knowledge (Laborde & Perrin-Glorian, 2005). In this milieu, the students can construct examples or counterexamples for a specific conditional to determine its truthfulness.

Social interaction

In the teaching experiment, students were incited to share ideas, analyse and criticize other's ideas. Having to formulate results obtained from an exploration makes students take distance from the actions themselves and these become an object of reflection. The propitiated social interaction affects the conceptualization process positively and leads students to focus their attention on a particular property and organize their thoughts as they argue. The teacher's role is decisive. Through *instructional conversations* (Tharp y Gallimore, 1988, cited in Forman, 1996) some of the actions carried out were: answering questions to help gain familiarity and comprehension of the geometric objects; accepting or rejecting formulated conjectures; presenting counterexamples when a conjecture is rejected; comparing the conjectures to determine whether they are expressing the same geometric fact; determining if the statement is correctly expressing the dependency relations. The teacher's role is fundamental and as important as the student's active role. Through *mathematical conversations* ideas were communicated, proposals presented, and commentaries and criticism made. The teacher not only organized the discussion but also monitored the development changing the course when necessary to manage the students' proposals, control the correct use of the elements of the axiomatic system, and institutionalize knowledge.

Didactic strategies

In what follows, we describe the strategies we designed to help students overcome the problematic performances identified.

Didactic Strategy A. The principal aim is to provide the students with mechanisms to grasp the structure of a conditional statement and convert the information provided into a reliable source. We hope students will overcome PP2 with this strategy. It consists of the following actions: 1) request complete identification of the hypothesis and thesis of the statements and their reformulation in the if-then form; 2) require the reformulation in specific terms, using the names given to the points involved in the figures which model the statement.

Didactic Strategy B. It was designed to address PP1 and focuses on acknowledging the difference between a conditional and it's reciprocal, converse and contra positive statements. With it, students are expected to understand why the use of propositions associated to a conditional does not always lead to a valid argument. The actions are: 1) examine if the use of a derived proposition leads to a valid argument; 2) promote making student's use and comprehension of a conditional explicit; 3) favour the use of the logic symbolization of the propositions involved in the argument.

Didactic Strategy C. Since students must formulate whatever they have discovered as conjectures in the if-then form, when exploring situations with dynamic geometry, it is not surprising to find in their productions a limited or distorted comprehension of a conditional. Strategy C was designed to help students overcome PP2 and PP3. The actions are: 1) request a description of all the actions carried out during the construction and exploration process with dynamic geometry and 2) identify the constructed properties and the resulting ones.

Didactic Strategy D. This strategy is centered on the deduction process needed to establish a step in a proof. It is related to the use of conditionals and to PP4. The actions are: 1) solicit the identification of definitions, postulates and theorems that have the same thesis as the statement which must be proven or a close relationship to it; that is, find a possible hypothesis for a geometric fact whose thesis is known, contrast it with the information already established, and determine if it's utility for the proof. 2) Analyse why an auxiliary construction is proposed in a proof, identifying the situation the construction generates and the links that can then be established within the situation to advance towards the culmination of the proof; 3) Request the analysis of whether all the conditions of the hypothesis of a specific geometric fact have been established so it can be used in the deductive process; 4) Require the identification of the previous steps in a proof used for each justification established, to reaffirm that it is a logical consequence of those steps.

RESEARCH INSTRUMENT

At the end of the semester, the students took a test that consisted of situations with a geometric context designed to detect whether the problematic performances were still an issue and decide if the strategies were effective. We present three of the tasks of the test and their objective.

Task 1 Rewrite the following statement in the if-then format so as to clearly state what must be proven: The diagonals of a rhombus bisect each other and are perpendicular.

With Task 1, we wanted to see whether students could recognize the antecedent and consequent in a statement not written in the if-then format, and if they used specific names to refer to the parts of the figure in their reformulation. That is we used it to determine whether *PP2* occurs and effectiveness of Strategy A.

Task 2 The proof Arthur provided for the following statement is given. Decide whether each step is correct. Explain your answer.

ABCD is a parallelogram with AD > AB. $\angle$A bisector intersects $\overline{BC}$ in G and $\angle$B bisector intersects $\overline{AD}$ in H. Prove that ABGH is a rhombus.

Statement	**Justification**
1. ABCD is a parallelogram, AD > AB, $\angle$A bisector intersects $\overline{BC}$ in G, $\angle$B bisector intersects $\overline{AD}$ in H	1. Given 
2. $\overline{AH} \parallel \overline{BG}$	2. Def. Parallelogram
3. $\angle 4 \cong \angle 5$, $\angle 2 \cong \angle 6$	3. Theorem: alternate interior angles of parallel lines
4. $\angle 1 \cong \angle 2$; $\angle 3 \cong \angle 4$	4. Def. Angle bisector
5. $\angle 3 \cong \angle 5$	5. Transitive property
6. $\overline{AB} \cong \overline{AH}$	6. Isosceles triangle theorem
7. ABGH is a rhombus	7. Def. has a pair of adjacent congruent sides

The purpose of Task 2 was to determine whether *PP4* occurs and if Strategy D was effective. Recognizing the theoretic elements that could be used to prove a quadrilateral is a rhombus and analysing the conditions required in the hypothesis to determine whether these have been included in previous steps of the deductive process constitute an exercise that must be carried out to establish if the definition mentioned in step 7 of the proof can be used. Through the students' analysis of the proof proposed, we recognize whether the students have the same performance. The definitions given for rhombus in class were: *parallelogram with a pair of congruent adjacent sides* and *quadrilateral with four congruent sides*. For the first one, proving that □*ABGH* is a parallelogram is lacking. For the second one, the congruency of the four sides has not been established. This problem evaluates the presence of *PP1* when the students being tested do not recognize (step 6) that the reciprocal of the Isosceles triangle theorem is being used and not the theorem itself; the justification is incorrect.

> **Task 3**. With dynamic geometry, explore the following situation. Study the relation between the type of quadrilateral and the property congruent diagonals. Give a brief description of the construction made and state a conjecture.

With Task 3 we wanted to determine whether, after exploring with dynamic geometry, students formulate a conjecture whose antecedent includes all the conditions given for the construction, and the consequent is the resulting property. We assumed that the students would use the dynamic geometry software with solvency, due to the constant use of Strategy C, and thus recognize the existent dependencies between the properties constructed and those that are a consequence of them, to formulate a conjecture that corresponded to the construction carried out.

The student's description of his construction, together with the posterior analysis that the teacher carried out of it and the constructed figure, were contrasted with the formulated conjecture to determine whether *PP2* or *PP3* had occurred. If the hypothesis of the conjecture did not correspond with the geometric conditions that the student deliberately constructed to explore the proposed situation, and these were reported as the thesis of the conjecture, there's evidence of *PP2b*. This type of exercise was done many times throughout the semester so we assumed that the students knew that they had to find all the dependencies that could be evidenced under dragging. If the student reports a generalization of the particular situation he had represented, *PP3a* is present. When this generalization is false, *PP2* occurs since falsity is the result of the absence of a condition either in the hypothesis or the thesis. Also, *PP3b* is evidenced if there is no correspondence between all the information that could be extracted from the exploration carried out with the representation and that which is established in the conjecture. Furthermore, if in the conjecture the students do not include the condition required in the problem in the hypothesis nor the thesis, we classified his answer as *PP2* or *PP3* because that condition must be, either the property constructed or obtained by dragging, or that which is consequence of the other properties established in the construction.

RESULTS

Only four of the eighteen students had difficulty in interpreting the statement given in Task 1 and therefore did not rewrite it as solicited. Additionally, ten students used specific names for the geometric figures, even though it was not solicited. This leads us to affirm that actions 1and 2 of Strategy A have a high probability of providing elements to overcome the difficulties associated with *PP2* and of affecting the way students express theorems and conjectures.

With respect to Strategy C, in Task 3 twenty conjectures were formulated by sixteen of the eighteen students. Five students suggested more than one conjecture. Only three of the conjectures did not correspond with the construction carried out, because two congruent bisecting segments were constructed and the endpoints joined to form a quadrilateral but the conjecture reported was *If a quadrilateral is a rectangle then the diagonals are congruent*. Six of the sixteen students generalized from their representation (*PP3a*) and, at the same time, did not report all the properties that the diagonals of their quadrilateral had (*PP3b*). Four of the students generalized correctly and two of these did not include all the conditions that the diagonals had (*PP3b*). Five students of the sixteen did not include, as part of their conjecture, that the diagonals are congruent (*PP2b*) but only one of them did not report having constructed congruent diagonals. We conclude, by examining their description of the construction, the figure they represented and the conjecture they proposed, that most students recognized the existent dependency relations between the properties constructed and those that arise as a consequence of them. But, twelve students did not report properties they could have perceived visually (*PP3b)*.

Strategies B and D refer to the use of conditionals in situations within a geometric context, specifically to proofs restricted to an axiomatic system. In Task 2, the students had to identify if the conditions required in the hypothesis of the Isosceles Triangle Theorem (action 3) had been obtained and included in previous steps (action 4), to discover the error made in step 6 of the proof. Twelve out of seventeen students that answered the problem did not notice the presence of *PP2* in step 6 – accepting as a justification the Isosceles Triangle Theorem when what was used was its reciprocal; therefore, they themselves incur in the same problem. Successfully discovering the error in step 7 can be considered as a positive manifestation of this strategy. *PP4* was evidenced in only six students.

CONCLUSION

Student participation is fostered through the strategies. The possibility of modifying their performance relies heavily on having the students participate genuinely in class, on their being critical with what others say, on providing ideas. The strategies take into account the social aspect, reason why mathematical discussions and instructional conversations are part of the actions. Undeniably, the use of dynamic geometry is essential to be able to attain understanding of the conditional itself as a mathematical object and of its use to report dependencies between properties and to deduce correct information from mathematical facts. It is the possibility of making ideas ostensive, manipulating the representations of geometric situations to study dependencies, accepting or rejecting statements that express relationships between properties that give students the elements to argue and contribute to the construction of knowledge. Both pillars, the milieu and social participation, are important, and articulated they propitiate the effectiveness of the strategies. We have an implicit assumption: the students will, on their own, use the strategies when faced with tasks similar to the ones presented in class.

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