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From an Operational to a Relational Conception of the Equal Sign. Third Graders' Developing Algebraic Thinking

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Abstract

We describe a teaching experiment about third grade students´ understanding of the equal sign and their initial forays into analyzing expressions. We used true/false and open number sentences in forms unfamiliar to the students to cause students to reconsider their conceptions of the equal sign. Our results suggest a sequence of three stages in the evolution of students' understanding of the equal sign with students progressing from a procedural/computational perspective to an analytic perspective. Our data show that as students deepened their conceptions about the equal sign they began to analyze expressions in ways that promoted algebraic thinking. We describe the essential elements of instruction that advanced this learning.

KEYWORDS: Algebraic thinking, Analyzing expressions, Arithmetic equations, Early-Algebra, Equal sign, Number sentences, Relational thinking.

As algebra has taken a more prominent role in mathematics education, many are advocating introducing children to it in primary school (e.g., Carraher, Schliemann, Brizuela, & Earnest, 2006; Kaput, 2000; National Council of Teacher of Mathematics, 2000), and open number sentences can be a good context to address this goal. Students are frequently introduced to equations by considering number sentences with an unknown where a figure or a line is used instead of a variable, as in $\_+4=5+7$ (Radford, 2000). Discussions about these equations and the properties they illustrate can help students to learn arithmetic with understanding and to develop a solid base for the later formal study of algebra by helping them to become aware of the structure underneath arithmetic (Carpenter, Franke, & Levi, 2003; Kieran, 1992; Resnick, 1992). Unfortunately students´ conceptions about the equal sign interfere with their ability to successfully solve and analyze equations (Carpenter et al., 2003; Molina & Ambrose, 2006). Our aim in this paper is to describe a teaching experiment in which children developed an understanding of the equal sign and began to analyze expressions in sophisticated ways.

Difficulties in Solving Number Sentences

When elementary students encounter the equal sign in arithmetic sentences they tend to perceive it as an operational symbol, that is a "do something" signal, and tend to react negatively when number sentences challenge their conceptions about this symbol as in sentences of the form $c=a+b$. Behr, Erlwanger, and Nichols (1980) observed that most six-year old students thought that such sentences were "backwards" and tended to change them to $c+a=b$ or $a+b=c$. Children also did not accept non-action sentences, that is sentences with no operational symbol (e.g., $3=3$) or operational symbols on both sides (e.g., $3+5=7+1$), and often changed them to action sentences, that is sentences with all the operations in one side of the equation. For example they changed $3+2=2+3$ to $3+2+2+3=10$ and $3=3$ to $3+0=3$ or $3-3=0$.

In studies about elementary students' answers to open sentences of the forms $a=a$, $c+a=b$, $a+b=c$ and $a+b=c+d$ (Falkner, Levi, & Carpenter 1999; Freiman & Lee, 2004; Kieran, 1981), students provided a variety of responses: repeating one of the numbers in the sentence, the sum or difference of two numbers of the sentence, the sum of all the numbers in the sentence, and the correct answer. In sentences of the form $a+b=\_+d$, students tended to answer the sum of $a+b$ or to write it as a string of operations.

These studies illustrate that children tend to read open number sentences from left to right and perform the computation as they go along. When children face unfamiliar number sentences, that is sentences different from the conventional form $a \pm b = c$, they have trouble interpreting the equal sign as a symbol representing equivalence. To successfully solve a problem such as $8 + 4 = \_ + 5$, students have to read the whole sentence before computing and need to recognize that both sides of the equation need to have the same sum. The equal sign needs to be interpreted as a relational symbol expressing equivalence.

Considering students' difficulties with the equal sign, their understanding of the concept of equality can be questioned; however, Schliemann, Carraher, Brizuela, and Jones (1998) and Falkner et al. (1999) have observed that students show a correct understanding of equality when considering concrete physical contexts or verbal word problems. The children's misinterpretations are linked to the use of the equal *symbol* rather than an understanding of the *concept* of equality (Carpenter et al., 2003; Falkner et al., 1999; Schifter, Monk, Russell, & Bastable, in press). Most studies about the equal sign (Behr et al., 1980; Falkner et al., 1999; Saenz-Ludlow & Walgamuth, 1998) have claimed that traditional curriculum does not promote a relational understanding of this symbol, mainly because of the repeated consideration of equations of the form a ± b = c throughout students' arithmetic learning. These misinterpretations may also be exacerbated by the linguistic convention of writing and reading from left to right (Rojano, 2002).

Recent research (Carpenter et al., 2003; Koehler, 2004; Saenz- Ludlow & Walgamuth, 1998) has shown that elementary students, even first graders, are capable of developing a relational understanding of the equal sign with suitable instruction going against previous claims about the existence of cognitive limitations in developing a relational meaning of the equal sign in the elementary grades (Kieran, 1981). Our study contributes and extends these findings by describing how children developed broader conceptions about this symbol as the result of classroom activities and discussion.

Analyzing Expressions

Carpenter et al. (2003) have used the term *relational thinking* to describe an approach to solving open number sentences. They illustrate this with the example of the problem $27 - 48 + 48 = \_$ (p. 32). Children who recognize that 48 and 48 are the same number and that addition and subtraction are inverse operations will conclude that the answer to the problem is 27 without having done any computation. Koehler (2004) more broadly refers to relational thinking as "*the many different relationships children recognize and construct between and within numbers, expressions, and operations* (p. 2)." We have chosen to adopt the term *analyzing expressions* for this kind of thinking to better distinguish it from the relational meaning of the equal sign as a symbol representing equivalence. We say that students use *analyzing expressions* when they approach number sentences, by focusing on arithmetic relations instead of computing. Students engaged in analyzing expressions employ their number sense and what Slavit (1999) called operation sense to consider arithmetic expressions from a structural perspective instead of a procedural one. Sentences have to be considered as wholes instead of as processes to do step by step. When students analyze expressions, they compare elements on one side of the equal sign to elements on the other side of the equal sign or they look for relations between elements on one side of the equation. For example, when considering the number sentence $8 + 4 = \_ + 5$ some students notice that both expressions include addition and that one of the addends, 4, on the left side is one less than the addend, 5, on the other side. Noticing this relation between these elements and having an implicit understanding of addition properties enables the student to solve this problem without having to perform the computations 8 plus 4 and 12 minus 5. Analyzing expressions by comparing elements on each side of the equal sign is the kind of thinking that students must do when solving algebraic equations in the form $3x + 7 = 2x + 18$. As students analyze expressions, they employ and deepen their operation sense which is fundamental to algebraic thinking (Slavit, 1999).

Analyzing expressions may form a good foundation for the formal study of algebra but little evidence is available to show that children are capable of doing so. Liebenberg, Sasman, and Olivier (1999) observed that most students were not able to solve open sentences without computing the answer due to a lack of knowledge about arithmetic operations and their properties. Kieran (1981) noted that "lack of closure" interfered with children's ability to solve sentences such as $8 + 4 = \_ + 5$ because nowhere is the value of each expression (12) represented. While Carpenter et al. (2003) provided anecdotal evidence that children were capable of overcoming the "lack of closure issue", they did not provide data as to the proportion of students able to do so nor the difficulties encountered in this process.

Research Design

Conjecture Driven Research Design

We applied the "conjecture-driven research design", which Confrey and Lachance (2000) propose for investigating new instructional strategies in classroom conditions and for analyzing different approaches to the content and the pedagogy of a set of mathematical topics. Our research method shared the features of design experimentation identified by Cobb and his colleagues (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The base of this design is a conjecture which guides the research process and is revised and further elaborated while the research is in progress. Multiple methods of collecting data are used for analyzing the effectiveness of the design.

Our research process was guided by the conjecture that elementary school students can develop their conceptions of the equal sign by discussing and working with true/false and open number sentences of forms that are unfamiliar to them. Critical to our design were the choice of tasks we provided to students. The tasks we used were number sentences, including true/false number sentences and open number sentences that were unfamiliar to the students. Most of the sentences that we chose could be solved by analyzing expressions. We adopted Simon's (1995) stance that challenging tasks stimulate student learning by triggering cognitive dissonance. We expected that the number sentences alone would not be sufficient to promote learning and that the teacher would have to "provoke disequilibrium" (Simon, 1995, p. 140). We chose to do this by introducing the "mathematician's interpretation" of equality which we expected would often differ from the children's. The general aim of this research work was to study students' thinking involved in solving number sentences, focusing on the understanding of the equal sign and analyzing expressions in the context of whole class activities and discussion. During the research process, consisting of five in-class interventions, we analyzed the evolution of students' conceptions about the equal sign, students' ways of and difficulties in solving number sentences, and students' development and use of analyzing expressions.

Subjects

We worked with a class of 18 third-grade students, who were between 9 and 10 years old, in an ethnically and linguistically diverse urban public school in California where 87% of the students were economically disadvantaged. Five students spoke a second language and two of them had significant difficulty understanding English. During the previous months we, as guest teachers, had worked with the class on a weekly basis doing a variety of mathematics activities. The official teacher was always present and sometimes collaborated with us in helping the students. In between our sessions, the teacher used a traditional textbook which emphasized computational practice.

Data Collection

We worked with the students over five sessions of variable duration (from twenty to fifty minutes) which took place during the students' regular school time (see Table 1 for schedule of activities). The first session took place two and a half months before the next one. The second, third and fourth sessions were fifteen days apart and the fifth session was two months later. The timing of the sessions was partly intended for helping our in-class interventions to induce a longer term effect on the students and partly accidental due to holiday breaks and our limited access to the classroom. We video recorded the first, second and fourth sessions. During the third session we took field notes. We also collected students' worksheets and took notes of the researchers' decisions and thoughts throughout the research process.

Table 1

Organization of the Sessions

Sessions	Date	Students in Class	Activities
1st	11-20-03	14	Written assessment (open sentences)
			Discussion
2nd	2-5-04	15	Written assessment (T/F sentences)
			Discussion
			Construction of non-action sentences
			Discussion
3rd	2-19-04	18	Discussion of T/F sentences
			Written assessment (T/F & open sentences)
4th	3-4-04	18	Discussion of T/F sentences
5th	5-13-04	15	Written assessment (open sentences)

We chose to work on the understanding of the equal sign with arithmetic sentences, instead of with manipulatives or word problems because our interest was on students' arithmetic knowledge in connection with arithmetic symbolism. The activities of the lessons were whole-class discussions and individual written assessments composed of open and true/false number sentences of varied forms, many of which were unfamiliar to the students ($a \pm b = c$, $c = a \pm b$, $a \pm b = c \pm d$ & $a \pm b = c \pm d \pm e$), specially designed to support the use and development of analyzing expressions. In most of the sentences we considered addition and subtraction expressions which could be easily solved by third grade students and which might stimulate them to engage in analyzing expressions (e.g., $14 + \_ = 13 + 4$, $15 + 2 = 15 + 3$). In the last two sessions, some sentences included bigger numbers (e.g., $103 + 205 = 105 + 203$) to provoke children to analyze expressions as a simpler way to address the sentences than doing the operations. We also asked the students to write their own non-action sentences with two terms on each side of the equal sign. (The collection of sentences considered in each session was reported elsewhere (Molina and Ambrose, 2006).)

Following Confrey and Lachance's (2000) recommendation for this research design, we considered the results of previous sessions as we designed the next ones. As a consequence, some of the considered sentences addressed previously detected difficulties, and some were taken from the students' work in previous sessions.

Results and Discussion

We studied the evolution of students´ understanding of the equal sign and students´ development of analyzing expressions during the five sessions. We noted students' progress through stages in understanding the equal sign, the difficulties they encountered in making sense of the equal sign and the classroom work that we did to advance their conceptions. Here we comment on the main results. First we describe our teaching approach.

In-Class Interaction and Teaching Approach

We hypothesized that all of the children would initially interpret the equal sign operationally and we planned to raise cognitive dissonance by challenging their interpretations. We expected that the combination of unfamiliar number sentences and our challenges to their interpretations would stimulate the children to think in new ways. We hoped that as individuals in the class began to develop a broader understanding of the equal sign, children's differing interpretations would continue to fuel dissonance that would stimulate assimilation and accommodation.

We began our first session by supplying the children with a set of open number sentences to solve independently. It was clear that these sentences were unfamiliar to the children because one asked, "what is that equal sign doing in the middle?" and another reported that he was not sure what he should do to solve them. After the children completed the sentences, they discussed their answers to two of the equations. In the first one, $8 + 4 = \_ + 5$, all of the students thought the answer was 12. We told them that "mathematicians" would disagree and highlighted the presence of 5 on the right side. The children suggested the answer was 17 (adding all the numbers), and we replied that "mathematicians" would still disagree. A student proposed to modify the sequence to $5 + 8 + 4 = 17$. Finally, we explained that "mathematicians" use the equal sign to show that the whole expression on one side is equal to the whole expression on the other side. A student then gave the answer 7. The discussion of the second problem, $14 + \_ = 13 + 4$, followed a similar pattern.

In this discussion we successfully created dissonance so that children's operational interpretations of the equal sign were challenged. We expected that this discussion might be sufficient for a few students to reconsider their interpretation but that many would need more opportunities to construct this understanding for themselves.

In future sessions we relied more heavily on the students' differing conceptions of the equal sign to fuel disagreements. For example, when discussing the sentence $2 + 2 + 2 = 3 + 3$, some students affirmed it was true and one explained *"it is true because 2 + 2 + 2 does equal 6 and so does 3 + 3".* Other students said that they thought it was false and explained *"I thought it should be 2 + 3 = 5,"* and "*I thought it was false because it has the equal sign in the middle".*

We continued to choose number sentences that would challenge the students' conceptions and that could fuel debate. In the fourth session we asked students whether $3 + 3 + 3 = 9 + 2 = 11$ was true or false. We chose this sentence because children often write sentences like this to keep track of their computation while solving multi-step problems. When some children claimed it was false, we challenged them by saying, "*3 plus 3 plus 3 is 9, and 9 plus 2 is 11. So isn't this true?*" This caused some children to reconsider their answers. One said, "I am not sure… It is in part true, and it also seems false." Others persisted in arguing that it was false because "*3 + 3 + 3 does not equal 11.*" When it was clear that the children could not resolve their differing interpretations, we told them that mathematicians would say the sentence was false.

It might have been preferable had we been able to guide the children to reconstruct their understanding of the equal sign by carefully choosing problems which challenged their thinking in the absence of having to introduce "mathematician's interpretations". Recently constructivists have drawn attention to the teachers' need to introduce ideas into discussions with students (Lobato, Clarke, & Ellis, 2005). We chose to do so in this case because we felt it would expedite children's learning. We were mindful that when a teacher does initiate ideas in this way, it is important for him/her to step back and to see how students take up these ideas rather than assume students understand. The data below demonstrate that students did develop relational understanding of the equal sign.

Stages in Understanding the Equal Sign

By analyzing students´ responses to the various sentences, we detected three stages in the evolution of students´ understanding of the equal sign, which we name stimulus for an answer, expression of an action and expression of equivalence. The first stage, *stimulus for an answer* (SA), refers to the interpretation of the equal sign as an operational symbol from left to right, that is as a command for giving the answer to the operations expressed on the left side of the equal sign. In this stage students tended to correctly solve sentences of the form a ± b = c but not sentences of the forms considered in this study. The students classified at this category mostly focused on the $a \pm b = c$ part of the longer sentence.

The following extracts from session 1 let us show this conception.

R: Can you tell me how you got this number (12 in the sentence $8 + 4 = \_ + 5$)?

S: Because eight plus four are twelve.

R: And, what happened with this five at the end?

S: That is also equal to something else.

When the student notes, "that is also equal to something else", we inferred he was thinking about chaining operations and perhaps thought he should write "= 17" at the end of the equation. However, he seems to be aware that this may not be the correct interpretation of the sentences.

R: How did you solve the last one ($\_ + 4 = 5 + 7$)? Why did you write a 1?

S: 'Cause I got a clue from the answer. Because it is four equal five, only four plus one equals five.

R: Ok… And what happened with this plus seven?

S: It is kind of difficult for me to understand this.

These comments illustrate how this kind of number sentence was unfamiliar to the child and he was unsure about how to respond to them.

The second stage, *expression of an action* (EA), refers to the interpretation of the equal sign as an operational symbol from left to right or right to left. In this stage students correctly solved sentences of the form c = a ± b, but gave wrong responses in sentences of the form $a + b = c + d$ such as 17 for the sentence $14 + \_ = 13 + 4$ (focusing on the $\_ = 13 + 4$ part of the equation) or 0 in $12 + 7 = 7 + \_$ (focusing on the $7 = 7 + \_$ part of the equation). In these cases they focused on a part of the equation that was in the $c = a \pm b$ form. In other sentences the extreme end of the equation was considered to be "the answer". For example, a student answered 5 to the sentence $12 + 7 = 7 + \_$, ignoring the 7 on the left side of the equal sign and focusing on $12 = 7 + \_$. Similarly another student answered 12 to the sentence $\_ + 4 = 5 + 7$, ignoring the 4 and focusing on $\_ = 5 + 7$. They also gave wrong responses like those associated with the "stimulus for an answer" conception. In this stage, students recognized the symmetric property of equality relation although they did not interpret the equal sign as the expression of equivalence. They continued to think about this symbol as a stimulus for an answer but recognized that the answer could be on either side of it. We hypothesized that they had become so used to sentences with two numbers that they tended to ignore a third number if it was present.

In the stimulus for an answer stage and the expression of an action stage, the children represented the "answer" to the expression in the equation. We attributed this to Kieran's (1981) "lack of closure" issue in which students expect to see the result of computation written down within an equation. These students considered number sentences as expressions of an action and not of a relation. The last stage, *expression of equivalence* (EE), refers to the interpretation of the equal sign as a relational symbol. In this stage students correctly solved sentences of all the considered forms.

In some sessions a few students were at what we called and "unstable" stage. These students did not have a consistent way of interpreting the sentences giving wrong responses related to various conceptions about the equal sign. For example in session 1, a student gave the following responses to problems: the response of the operations on the left side (12 to the sentence $8 + 4 = \_ + 5$), the response of the operation on the right side (12 to the sentence $\_ + 4 = 5 + 7$) and correctly solved the sentence $12 + 7 = 7 + \_$. This student seemed to be interpreting the equal sign differently from one sentence to the other reflecting the uncertainty mentioned earlier.

Students' progress through the stages. In Table 2 we show how students advanced in their understanding of the equal sign moving through these three stages .We do not refer to session 4 as we did not collect individual information on that day.

Table 2

Evolution of Students´ Conceptions about the Equal Sign

Conceptions about the Equal Sign	Sentences Correctly Solved	1st Session N = 14	2nd Session N = 15	3rd Session N = 18	5th Session Final Test N = 15
Stimulus for an answer	$a \pm b = c$	8	5	0	1
Expression of an action	$a \pm b = c$ & $c = a \pm b$	4	5	3	1
Expression of equivalence	$a \pm b = c$ & $c = a \pm b$ & $a \pm b = c \pm d$	0	3	12	12
Unstable		2	2	3	1

To clarify how students' individual understanding of the equal sign evolved, we show the conceptions for each student for each session in Table 3. Eight students followed the evolution previously outlined which we called the "Expected" trajectory. Another five students directly evolved from the conception *stimulus for an answer* to *expression of equivalence*, not showing the conception *expression of an action*. We called this the "Skip" trajectory because these students did not pass through a period of having the *expression of action* conception.

Table 3

Evolution of Students' Understanding of the Equal Sign through the Sessions

Student	Session 1st	2nd	3rd	5th	Developmental Trajectory
1					Expected
2					Expected
3					Skip
4					Skip
5	US				Expected
6					Regressed
7					Expected
8					Skip
9	US				Expected
10		US			Expected
11					Skip
12					Expected
13			US		Skip
14					Regressed
15					Expected
16		US		US	Regressed
17					Expected
18			US		Special Case

= stimulus for an action,

= expression of action,

= expression of equivalence,

US = unstable.

Students 6, 14 and 16 showed an evolution which goes against the SA-EA-EE order of development. Student 6 regressed in her understanding from session 1 to 2: initially she correctly solved the sentence $\_ = 25 - 12$ but later, in session 2, thought that $10 = 4 + 6$ was false. At that moment, this student seemed to be at a transitional stage in her thinking, just beginning to appreciate the symmetric property of the equal sign. Student 14 showed the conception *expression of equivalence* in session 3 but later incorrectly answered at least five of the seven sentences in session 5 showing regression in his understanding. We hypothesize that he needed to encounter sentences of varied forms more frequently to be able to consolidate his understanding. It similarly occurred with student 16, who after having correctly solved all the sentences in session 3, did not show a clear conception in session 5. The other student only attended one session so we cannot make claims about his development.

Table 3 illustrates that three students quickly adopted the *expression of equivalence* conception, developing it after one class discussion of the meanings of the equal sign. Most students developed the *expression of equivalence* conception by the end of the third session. Three students required all four days of instruction to advance to the stage of *expression of equivalence* and two students regressed between the third and fifth sessions.

Difficulties Encountered Understanding the Equal Sign

The discussion in session 1 was not sufficient for helping most students to adopt a relational understanding of this symbol. Only three students did so. Most students needed to work on the sentences through the different sessions and in different contexts. The use of the equal sign in non-action sentences seemed unnatural to them, and we discussed that they probably had never seen sentences like this before.

By varying where the unknown was in a four-term equation, we were able to determine the nature of students' conceptions, the consistency of their application of them and their behaviour when their conception did not apply. We found that their previous experiences with the equal sign had fostered the kinds of conceptions identified by previous researchers (Behr et al., 1980; Falkner et al., 1999; Kieran, 1981).

Students' answers to the non-action sentences of session 1 were the correct response, repeating one of the terms, operating on all the numbers in the sentence and ignoring one of the terms. When one of the terms of the sentence was ignored, some students changed the operation or the order of the terms in the sentence. For example a student answered 2 to $\_ + 4 = 5 + 7$ ignoring the 4 and subtracting 7 minus 5. Another student answered 40 to $\_ + 0 = 30 - 10$ explaining that 40 minus 10 is 30.

Most of the students' errors were due to their ignoring the final term in the sentence. When this was impossible to do as in $12 + 7 = 7 + \_$, students provided a variety of incorrect answers. Students' answers to some of the problems show that they were capable of finding missing addends or subtrahends. Therefore their difficulties in solving the sentences were due to their limited understanding of the equal sign.

When students were solving the open sentences in session 1, most of them proceeded immediately to do some computation without even looking at the whole sentence. They were focused on calculating and getting the answer, giving no attention to the sentence as a whole. We hypothesize that this behaviour is a consequence of the strong orientation to computation which dominates arithmetic, especially in the earlier grades (Kieran, 1989; Liebenberg et al., 1999). This computational mindset also interferes with students' work in solving algebraic equations where analysis of the whole equation is required before manipulation of it.

Some answers revealed that students were unsure of what to do, for example when students ignored one of the terms and changed the operation or the order of the terms or when they repeated one of the numbers in the sentence. In session 1 we observed that simply to have the answer on the left side of the equal sign confused them and only three of the thirteen students responded 13 to the sentence $\_ = 25 - 12$. In later sessions while more students successfully solved the considered open sentences, those that continued to make errors tended to ignore the final term.

As Lindvall and Ibarra (1980) observed, for many students the sentences were not expressions of relations but a list of numbers and operational symbols, and students applied the operations to the numbers as they considered it possible or most convenient. We find a clear example of this statement in session 1 when we asked a student how he got the answer 26 to the sentence $12 + 7 = 7 + \_$. He explained *"Cause twelve plus seven is nineteen and then there is equal seven and then there is a plus again, but if we move this (7) here, it is twelve plus seven plus seven".*

During session 2 the students analyzed true/false sentences. Important difficulties occurred because of the middle position of the equal sign in most of the sentences and because of the lack of operation in sentences of the form a=a. Students explained that they preferred to see the equal sign "at the end" of the sentence and asked for specific explanations about a=a sentences. Nine of the fifteen students modified the sentence $3 = 3$ writing $3 + 0 = 3$, $0 + 3 = 3$ and $3 + 3 = 6$. They also reacted negatively to sentences of the form $c = a \pm b$ saying that they were false for being backwards and changed them (e.g., they changed $10 = 4 + 6$ to $4 + 6 = 10$ or $6 + 4 = 10$).

Some students incorrectly used the equal sign to express a string of operations as in the discussion of the sentence $37 + 14 = 38 + 13$ where a student proposed to change it to $37 + 14 = 51 + 16 = 77$. Even after discussing the sentence $3 + 3 + 3 = 9 + 2 = 11$ a student wrote the sentence 55 = 5 x 11 = 2 + 9 = 3 x 3 and accepted using the equal sign for chaining operations in other sentences. This is an example of the special difficulty that students encounter in understanding the unsuitability of this use of the equal sign. For them it makes sense to chain the operations by using the equal sign probably because it corresponds with the way these sequences of operations are stated verbally, and the fact that they have written sentences like this to convey their thinking on multiple step problems. Addressing this chain of operations misconception during instruction appears to be especially important because of its durability.

Classroom Work Promoting Advances in Children's Conceptions

Engaging students in discussions in which they had to defend their opinions was critical to the development of their understanding of the equal sign. Students confronted each others' answers which led them to encounter different ways of interpreting the same sentence. This exchange and the variety of sentences caused the students some cognitive dissonance which drove further learning. This dissonance was first produced during the discussion in session 1 when we explained the relational meaning of the equal sign. Considering true/false sentences was a good way to force students to look at the sentences as a whole while challenging their conceptions of the equal sign. It also helped to restrain their computational tendency. The consideration of sentences of the form $a = a$, where no operational understanding is possible also served to provoke disequilibrium.

Data in Table 3 suggest that the period between the assessments at the start of session 2 and at the end of session 3 was critical to students' learning. We suspect that the active use of the equal sign in the construction of their own true number sentences with four terms in session 2, fostered growth for many students as they tried to apply their conceptions of the equals sign. This work allowed them to utilize their emergent understanding of the equal sign. It also required that they create their own sentences with four numbers and two operations which interfered with their assumption that open number sentences contained two numbers and one operation.

In this activity, some students required assistance in including a fourth term, and most finally constructed several non-action sentences. Three students wrote sentences with three or four equivalent expressions such as $10 + 11 = 21 = 20 + 1$ or $25 + 10 = 35 = 30 + 5 = 35$. We hypothesize that this type of sentence helped some of the students' transition to the *expression of equivalence* conception because it allowed them to represent all of the steps in their thinking. They could compute an "answer" to the first expression, represent that answer, and then derive another expression equivalent to their "answer." This accommodated their *stimulus for an answer* conception and their emergent *expression of action* conception and could be considered a way to put an equation of the form $a + b = c$ together with an equation of the form $c = a + b$. We used a sentence of this form in the class discussion during session 3 (10 x 10 =100 = 90 + 10) which may have helped the students' transition to the *expression of equivalence* conception.

In summary, all of the students in the class began the study with a limited understanding of the equal sign. For the majority of the class, the activities and discussions enabled them to advance to the stage of interpreting this symbol as an expression of equivalence. Some students passed through a phase of considering the equal sign to be an expression of action. Recognizing the expression of action stage of children's development and the value of number sentences with three expressions should be helpful to those planning instruction to address students' conceptions about the equal sign.

Development of Analyzing Expressions

Beginning in session 1 we looked for evidence of students' analyzing expressions and considered sentences designed to promote this approach. But it was after session 2 that we started to explicitly promote the use of analyzing expressions by asking the students if they could solve the sentences without doing all the arithmetic. We did not promote the learning of concrete relational strategies but the development of a habit of looking for relations, trying to help students to make explicit and apply the knowledge of structural properties of arithmetic which they had from their previous experience with arithmetic. We encouraged children to begin analyzing expressions by asking them to look for different ways of solving the same sentence and by encouraging them to develop explanations based on relations. Some students were motivated to analyze expressions because they wanted to too quickly get an answer and be the first ones to raise their hand.

We observed that eleven of the eighteen students used analyzing expressions at some point in the five sessions (see Table 4). Their explanations were based on the observation of relations between the terms in the sentences instead of in computing the sequence of operations of each member. In addition, other students constructed sentences or solved them in ways which made us suspect a possible use of analyzing expressions; however, they did not articulate this thinking. For example a student wrote several sentences of the structure $a - b = (a + 1) - (b + 1)$ (see Figure 1) which suggest that she may have considered relations between the numbers on either side of the equal sign and/or considered relations between the numbers from one equation to the next. She explained that all of the problems had an answer of 6.

Table 4

Number of Students who Gave Evidences of Analyzing Expressions in Each Session

1st Session N = 14	2nd Session N = 15	3rd Session N = 18	4th Session N = 18	5th Session N = 15
1	1	2	6	7

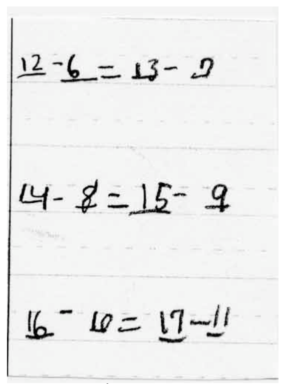

Figure 1

The first evidence of analyzing expressions was detected in session 1 when a student explained his answer (3) to the sentence $14 + \_ = 13 + 4$ referring to the compensation relation of addition: *"I looked to this side and…they shifted them [...] the three and the four"*. In session 2 another student explained that $34 = 34 + 12$ was false *"because thirty-four plus twelve would be more than thirty-four"*, reasoning without doing the operation by comparing the expressions 34 and 34 + 12.

This approach became more frequent in the next sessions although students also provided explanations based on the computation of the operations. In all but one of the sentences of session 4 students gave various explanations based on analysing expressions. In the sentence $238 + 49 = \_ + 40 + 9$, from the assessment in session 5, seven of the twelve students who had time to answer this question correctly solved it. Four of these students gave clear explanations which showed the use of analysing expressions: *"I realised that the number was 49 and 40 + 9 = 49 so I added 238"*, *"Split the 49 in half into 40 and 9. I did not add 238 and 49"* and *"I thought about the forty and the nine and I thought the zero didn't count so it's forty and nine and* *it's equal".* The other three students gave confusing explanations but we assume that they solved the sentence by using analyzing expressions because they did not subtract and we cannot imagine another way in which they could have solved it. Therefore, almost half of the students used analyzing expressions in at least one sentence.

In students' explanations based on analyzing expressions we identified the use of three types of relations (a) sameness of numbers or operations (e.g., in $7 + 15 = 100 + 100$ some students first noticed that the operations involved in both members were the same), (b) the reflexive property of the equality relation (e.g., in $20 + 20 = 20 + 20$ students reasoned that it was true because all the numbers were the same) and (c) the following properties of operations:

1. Commutative property of addition (e.g., "*12 + 11 = 11 + 12 is true because it has got the same numbers. 12 is in the front and later in the back and 11 is in the back and later in the front", "It is true because they changed the order of the numbers*").
2. Compensation relation of addition (e.g., *"51 + 51 = 50 + 52 is true because if you move the one from the fifty-one to the other fifty-one you get fifty plus fifty-two").*
3. Associative property of addition (e.g., *"103 + 205 = 105 + 203 is true because one hundreds and three plus two hundreds and five is equal to eight, and one hundreds and five plus two hundreds and three are eight, and there are two eights matching").*
4. Constant difference property of subtraction (e.g., "*12 – 7 = 13 – 8 is true because they added one to the seven and one to the twelve").*
5. Composition and decomposition relations (e.g., *"20 + 15 = 20 + 10 + 5 is true because ten plus five is fifteen").*
6. "Magnitude relations", that is relations based on the effect of operations on numbers and the relative magnitude of numbers (e.g., "*7 + 15 = 100 + 100 is false because seven plus fifteen is small and one hundred plus one hundred is two hundreds", "seven plus fifteen is not even one hundreds").*
7. Inverse relation of addition and subtraction (e.g., "*27 + 48 – 48 = 27 is true because there is a plus forty-eight and a minus forty-eight… and that is going to be zero"*).

While someone might argue that these sentences are somewhat obvious and do not require much insight, we categorized the students' statements as analyzing expressions because they were not computing to obtain answers and instead were looking at relations between or within expressions. The students had resisted the urge to compute and were instead looking at the sentence as a whole. We hypothesize that as children continued to use and articulate these relations, they could begin to discuss the generalizability of each. For most of these relations, it would be impossible for children to recognize and utilize them if they did not understand the equal sign. Carpenter et al. (2003) assert that instruction can proceed from exploring the equal sign through analyzing expressions to generalizing about properties. This study establishes the viability of the first part of the trajectory. We look forward to researching the degree to which students engage in generalizing.

Conclusions

This study confirmed and further explored the difficulties that third grade students encounter in understanding the equal sign. Our data support Carpenter et al.'s (2003) assertion that solving and discussing true/false and open number sentences is a fruitful venue for addressing students' understanding of the equal sign. We found that different kinds of number sentences reveal different conceptions and challenge children to reconsider their interpretations of the equal sign. Our data suggest a sequence of stages through which the students' understanding of the equal sign seems to evolve. We were particularly interested to find that some students resolved Kieran's (1981) "lack of closure" issue by creating sentences with three expressions, one of which was a single value. In this way they addressed their need to see "the answer" in a way that did not violate the meaning of the equal sign in this type of sentence. Future instruction might take advantage of this bridge to support the transition from *expression of action* to *expression of equivalence*.

The activities considered, especially the discussions of true/false number sentences, supported the development of analyzing expressions; of particular importance to this development was helping the students to overcome the urge to compute when they saw an equal sign and instead to look at the sentence as a whole. The nature of the sentences we provided engaged the students in making the transition from a computational to a structural/analytical perspective, a transition that is all too rare throughout most students' arithmetic learning. Contrary to the supposition of the existence of cognitive limitations in young students' development of relational understanding of the equal sign and analyzing expressions, we have shown that third grade students are able to develop this understanding and some of them use analyzing expressions for solving the sentences when the use of different strategies is encouraged.

From our view, as well as from other researchers' (Carpenter et al., 2003; Koehler, 2004) analyzing expressions can help the development of a semantic learning of arithmetic which as Booth (1989) affirms, is one of the prerequisites for developing the ability to understand and manipulate the notational conventions of algebra. In addition, it helps to lessen the frequent operational approach to teaching arithmetic which is considered one of the main causes of students' lack of awareness of the structure of mathematics operations and their properties and their difficulties in the learning of algebra (Liebenberg et al., 1999). We propose that a focus on analyzing expressions fostered an algebraic disposition towards arithmetic rather than a computational disposition.

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