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Enriching Arithmetic Learning by Promoting Relational Thinking[1]

Following the Early Algebra proposal of integrating algebraic thinking in elementary mathematics, we focus our attention on the development of relational thinking within arithmetic contexts. This thinking encourages exploring relations between numbers and between operations promoting a meaningful learning of Arithmetic and the development of a good foundation for the formal study of Algebra. In this document we present data from a study of third grade students who developed relational thinking in the context of discussions about number sentences.

1. The Early Algebra Reform

The traditional teaching of Algebra is widely criticized by numerous researchers (Mason, Davis, Love & Schoenfeld, according to Lee, in press; Kaput 1999; Booth, 1989). The international critique is based on the high number of students who fail in this area and stop studying mathematics, the absence of meaning in students' algebraic learning and the lack of connection between Algebra and other mathematical areas. According to Martin Kindt (1980, cited by Van Reeuwijk, in press), three of the big problems of algebra teaching are the lack of attention to generalization and reasoning, the tendency to jump too quickly to the formal study of Algebra, and the lack of clarity on for what and for whom Algebra is useful. The dissatisfaction with the teaching of Algebra, the recognition of the importance of algebraic habit of minds, and the necessity of making the study of Algebra more accessible to all students have led to mathematics educators to look for more effective ways of teaching Algebra. In the last decade there have been various proposals in this direction including: basing algebra learning on problem solving, using technology to engage students in algebraic thinking and an emphasis in strengthening arithmetic abilities (Freiman & Lee, 2004). Nowadays, researchers are considering a wider conception of Algebra to be integrated in the curriculum, beginning in the elementary grades, with an active and exploratory methodology (Kaput, 1999). Algebra is no longer thought of as a subject but as a way of thinking and acting on mathematics objects, structures and situations.

The NCTM (2000) and numerous researchers support this proposal, known as Early Algebra, in which Algebra is thought to have the potential to enrich mathematical activity and to serve as a guide for promoting learning with understanding (Carpenter, Franke & Levi, 2003; Carraher, Schliemann & Brizuela, 2000; Kaput, 1999; Bastable & Schifter, in press). In line with this

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proposal, teachers should foster algebraic thinking while teaching other mathematical concepts by helping students to pay attention to mathematical properties, relations and patterns. According to Blanton and Kaput (2003), teachers should create a classroom culture that values students' modeling, exploring, arguing, predicting, conjecturing, testing their ideas, and practicing computational skills. These interactions can be provoked by asking students questions that help them to verbalize and extend their thinking, such as what were you thinking? Can you think of a different way to solve this problem? How do you know this is true? Does this always work?

Supporting this reform, numerous researchers (Carpenter, Franke & Levi, 2003; Carraher, Schliemann & Brizuela, 2000; Bastable & Schifter, in press) have studied various aspects of Algebra and its role in elementary mathematics activities. These studies show that elementary students are able to address this new challenge. "When teaching is based on students' mathematical ideas and promotes their mathematical curiosity, students tend to show algebraic ways of thinking in arithmetic, geometric or measuring contexts" (Bastable & Schifter, in press, p. 2). The general aim is to promote advanced mathematical thinking in keeping with elementary students' capacities.

1.1 What is Algebra in Early Algebra?

The proposal of introducing Algebra in the curriculum from the earlier grades makes necessary a wide definition of what should be considered in school algebra. Numerous researchers have tried to answer this question in many ways without reaching any consensus. Algebra has been considered a language, a way of thinking, a tool, an activity and the generalization of Arithmetic (Lee, in press).

Kaput (1999) recognized the impossibility of listing all components of Algebra, or Early Algebra, and distinguished a few of them:

- Generalization of patterns and relations (specially generalized Arithmetic and qualitative thinking).
- Functional thinking.
- Modeling.
- Syntactically guided manipulation of formalisms.
- The study of structures.

This broad definition allows integrating Algebra in elementary curriculum from multiple approaches and provides a general idea of the different aspects of Algebra which current research is focusing on.

2. Early Algebra and Arithmetic. Relational thinking.

Various researchers (Warren, 2004; Carpenter, Franke & Levi, 2003; Carraher, Schliemann y Brizuela, 2000) have addressed the Early Algebra proposal within the teaching of Arithmetic. They claim that the separation of Algebra and Arithmetic accentuates and prolongs students' difficulties and recommend integrating both in the curriculum as soon as possible.

Traditionally, Algebra is introduced after Arithmetic when students are supposed to have acquired the necessary arithmetic abilities and knowledge about the structure of the operations. Mathematical relations are hypothetically learned by students during Arithmetic (Booth, 1989). This approach relies on an inductive generation of knowledge since fundamental mathematical properties such as the associative and commutative properties and other derived relations (e.g. $a + b = (a - c) + (b + c)$) underlie the arithmetical algorithms as well as most students' computational strategies. However, research shows that students have a poor understanding of the meaning of relations and mathematical structures (Booth, 1989; Warren, 2004; Kieran, 1989). "…a major part of students' difficulties in algebra stems precisely from their lack of understanding of arithmetical relations. The ability to work meaningfully in Algebra, and thereby handle the notational conventions with ease, requires that students first develop a semantic understanding of Arithmetic" (Booth, 1989, p. 58). Understanding and using the abstract properties of operations, i.e. understanding the internal structure of operations and the relations between them, together with the capacity of relating those operations with real situations, is one of the main aspects of understanding arithmetic operations (Dickson, Brown & Gibson, 1991).

Considering these claims, in an attempt to help student in the development of a semantic understanding of Arithmetic, we propose to promote the development and use of relational thinking while students are learning Arithmetic.

2.1 What do we mean by relational thinking?

We say that a person thinks relationally or uses relational thinking when he/she examines two or more mathematical ideas or objects alternatively looking for connections between them and, analyzes or uses those relationships in order to solve a problem, make a decision, or learn more about the situation or concepts involved. In the context of Arithmetic this term has also been referred as "the many different relationships children recognize and construct between and within numbers, expressions, and operations" (Koehler, 2004, p.2). This thinking is highly important in mathematics as "many fundamental mathematical ideas include relations between different representations of numbers and of operations between them" (Carpenter, Franke & Levi, 2003, p. 38), and between other mathematical objects. Because establishing relations among mathematical ideas or concepts is considered at the heart of understanding (Hiebert & Carpenter, 1992), relational thinking has the potential to help students to develop a deep understanding of Arithmetic upon which future abstractions in Algebra can be based (Molina, 2005; Koehler, 2004, Carpenter, Franke & Levi, 2003).

Relational thinking can be developed in a variety of activities by helping students to pay attention to relations between the operations and numbers involved and keeping the focus away from computing and getting the answer (Molina, 2005). Specifically, this thinking can be promoted in the context of number sentences where concrete relations can be represented in order to help students to focus their attention on them (Carpenter, Franke & Levi, 2003;

Koehler, 2004). For example, in the sentence 27 + 48 – 48 = ___, students might recognize that adding and then subtracting forty-eight will leave twenty-seven unaffected, therefore avoiding computation. In the case of the sentence 12 + 7 = __ + 12, students may deduce the answer by observing that the order of the addends has been inverted, instead of adding 12 and 7 and then solving the problem 19 = __ + 12. Similarly, the sentence 8 + 4 = __ + 5 can be solved by noticing that five is one more than four, so the unknown number has to be one less than eight. This thinking helps to minimize the computation and makes students think in terms of properties of operations, manipulation of numeric expressions and how this manipulation affects to the expressions (Molina, 2005; Koehler, 2004). This knowledge not only helps to develop arithmetic knowledge but also to develop fluidity in computation (Carpenter, Franke & Levi, 2003).

To engage in this type of thinking, students need to treat equations as objects to be analyzed rather than a process in which to engage. The duality process/object as well as the ability to move back and forth between these two modes of thinking has been widely studied in broader contexts by Sfard (1991) who identified it as fundamental to an understanding of function.

An important difference between the above examples is the understanding of the meaning of the equal sign needed for solving the sentence. While solving the first sentence does not require a broad understanding of the equal sign (because all the computation takes place on the left), the other sentences require understanding that the equal sign represents equivalence between two expressions. These differences must be taken into account when introducing students to number sentences, because many students tend to assume that the equal sign is a signal to compute rather than a symbol representing equivalence (see Molina & Ambrose, in press).

Focusing on addition and subtraction we have identified eight main relations and properties which can be addressed in the context of number sentences:

- The complementary relation of addition and subtraction.
- The commutative property of addition.
- The “compensation” relation of addition: the result of a sum does not change when a number is added to an addend and subtracted from another addend.
- The “compensation” relation of subtraction: the result of a subtraction does not change when the same number is added or subtracted from both terms.
- Adding cero leaves any number unaffected.
- When cero is subtracted to a number, the number does not change.
- Every number minus itself is cero.
- The associative property of addition.

Specifically the development of knowledge about place value and the various decompositions of a number, or an expression, can be addressed by considering number sentences such as 34 + 15 = 30 + 4 + 10 + 5 or 7 + 7 + 6 = 14 + 6. The decomposition of a number in addends is useful for deducing number facts from previously known facts (i.e. using knowledge about doubles). These relations and properties are usually implicit in students’ computations but are made more explicit when using relational thinking and students' solution strategies are discussed with the whole class. Number sentences allow students to share their thinking about basic and important mathematical ideas.

2.2 Previous studies about the development of relational thinking

Traditional instruction has failed to promote this kind of thinking (Liebenberg, Sasman & Olivier, 1999). These authors observed that most students were not able to solve open sentences without computing the answer due to a lack of knowledge about arithmetic operations and their properties. Kieran (1981) and Collis (1974, cited by Kieran, 1981) have claimed that students younger than thirteen are not capable of using relational thinking in solving number sentences because they need to see the answer after the equal sign. They relate this limitation to the difficulty involved in considering expressions as objects (as a whole) and not as a sequence of operations to be carried out.

In more recent studies (Carpenter, Franke & Levi, 2003; Koehler, 2004) this more sophisticated approach to solving number sentences has been observed in some elementary grade students. In Koehler' study, five low performing third and second graders "were able to develop knowledge of relationships between numbers, operations and expressions to support the learning of multiplication facts with understanding" (p. 55). This study shows that students do not need to master computational abilities to participate in more advanced mathematical activities such as the use of relational thinking. In fact, Koehler found that relational thinking helped to promote more sophisticated computation. Carpenter, Levi and Falkner (2003), following Davis's (1964) work, suggested open sentences and true/false sentences be used for working on the understanding of the equal sign and the development of relational thinking. Number sentences afford the opportunity to have very explicit discussions about relations between numbers and operations (Koehler, 2004).

Our study provides further evidence of the capacity of third grade students of developing relational thinking as well as examples of the kind of verbalization students may provide when starting to think relationally.

3. Our study

3.1 Methodology

We worked with a class of twenty third grade students (eighteen of them participants in this study) over five sessions which took place during the students' regular school time in their regular classroom. Students were used to working with us on a weekly basis doing a variety of mathematics activities. The classroom teacher was always present and sometimes collaborated with us in helping the students. The five lessons happened between December and May with several weeks in between each lesson. In order to properly analyze students' work and verbalizations, we collected students' worksheets, took notes during the lessons and we video recorded the main sessions.

During these five lessons we focused on solving number sentences. During the first two lessons, we helped students to develop their understanding of the equal sign by proposing and discussing open and true/false sentences of the forms a = a, a ± b = c, c = a ± b and a ± b = c ± d (See Molina & Ambrose, in press, for further details). In the next three lessons we worked on promoting students' development of relational thinking by considering true/false and open number sentences which could be easily solved by noticing certain patterns in the sentences (i.e. 34 = 34 + 12, 12 + 11 = 11 + 12, 51 + 51 = 50 + 52, 15 + 2 = 15 + 3, 34 + 28 = 30 + 20 + 4 + 8 …). The number sentences were not presented to students in groups that focused on a particular relation but were all mixed up. In this way we expected to help students to develop a habit of looking for relations not just the learning of particular relational strategies such as "adding 9 is the same as adding 10 and subtracting 1". We asked the students to say if the sentences were true or false and to correct the false ones. While discussing the students' answers, in an attempt to promote the use and verbalization of relational thinking, we asked the students if they could solve the sentences without doing the arithmetic.

The sentences were designed considering Carpenter, Franke and Levi (2003)'s recommendations. Specifically, we used open number sentences to evaluate students' understanding of the meaning of the equal sign and true/false sentences for encouraging students' verbalizations of their thinking and challenging their understanding. The true/false sentences served to break students' computational attitude and forced them to look to the whole sentence. We tried to avoid any difficulty not related to the understanding of the equal sign, so in most of the sentences we considered additions and subtractions which could be easily solved by most third grade students. In the last two sessions, some sentences included bigger numbers (e. g. 103 + 205 = 105 + 203) in order to provoke the use of relational thinking as a simpler way to address the sentences than doing the operations.

3.2 Results about the development of relational thinking

We analyze students' development of relational thinking by focusing on their verbalizations during the discussions of true/false and open number sentences as well as the sentences that they wrote.

We observe that eleven of the eighteen students gave explanations referring to the use of relational thinking at some point during the five sessions. In addition, other students constructed number sentences or solved the sentences in ways which make us suspect a possible use of relational thinking; however they did not make their thinking explicit. Initially a few individuals noticed relations within a sentence but, on later sessions, most students got engaged in observing relations and began to appreciate the benefits of this strategy for judging and solving number sentences.

The first evidence of relational thinking was detected on the first session when a student explained his answer (3) to the sentence 14 + _ = 13 + 4 saying that *"I looked to this side and…they shifted them […] the three and the four"*. He explained his answer by referring to the compensation relation of addition. Later on the second session another student explained that the sentence 34 = 34 + 12 was false *"because thirty-four plus twelve would be more than thirty-four"*. This student justified the falseness without doing the operation by comparing the expressions 34 and 34 +12.

Later, during session 3, more students verbalized this type of thinking giving explanations such as:

7 + 15 = 100 + 100 is false *"because seven plus fifteen is small and one hundred plus one hundred is two hundreds", "seven plus fifteen is not even one hundreds"*. In these explanations students recognized the difference in magnitude between the numbers on both sides of the equal sign and did not need to compute to reason the falseness of the sentence.

51 + 51 = 50 + 52 is true *"because if you move the one from the fifty-one to the other fifty-one you get fifty plus fifty-two"*. This student noticed the compensation of the expressions on both sides of the sentence.

15 + 2 = 15 + 3 is false *"because three is bigger than two"*. This student noticed the repetition of fifteen on both sides and deduced her answer by using her understanding of the operation of addition.

12 + 11 = 11 + 12 is true because *"it has got the same numbers. 12 is in the front and later in the back and 11 is in the back and later in the front"*, "*It is true because they changed the order of the numbers"*.

20 + 20 = 20 + 20 is true *"because they are the same numbers"*.

On these two last sentences students did not perform any computation for solving them; they compared the numbers on each side. However, we did not further explore these explanations and we do not know if the students would have similarly affirmed that the followings sentences are true, 21 + 21 = 12 + 12 and 34 – 15 = 15 – 34, "because they have the same numbers". At this point in their learning we were satisfied that students were looking at the sentence as a whole rather than performing computations. Had we more time to work with the students we would have begun to explore the limits of the commutative property.

During session 4, the students gave explanations based on relational thinking in all but one of the eight sentences considered (37 + 23 = 142, 27 + 48 – 48 = 27, 34 + 28 = 30 + 20 + 4 + 8, 76 = 50 – 14, 4 x 5 = 5 +5 +5 + 4, 20 + 15 = 20 + 10 + 5, 103 + 205 = 105 + 203 y 12 – 7 = 13 – 8). For example a student explained: 27 + 48 – 48 = 27 is true *"because there is a plus forty-eight and a minus forty-eight… and that is going to be zero"*, recognizing the inverse relation between addition and subtraction. Others students explained: 12 – 7 = 13 – 8 is true *"because they added one to the seven and one to the twelve"*, 37 + 23 = 142 *"is false because it has to be small"*, 76 = 50 – 14 is false because *"fifty isn't bigger than seventy-six and if we subtract, it can not be bigger"* and 20 + 15 = 20 + 10 + 5 *"is true because ten plus five is fifteen"*. These verbalizations showed the recognition of relations between the expressions on both sides of the equal sign and the use of knowledge about the operations of addition and subtraction.

In all the sentences students also gave explanations based on the computation of the operations. In addition, students gave some confused explanations, such as *"The way I found out my answer to numbers even by doing it the other way"*. Some students encountered more difficulties in explaining their thinking when it did refer to relationships instead of concrete operations. In those cases the researcher guiding the class helped the students to explain what they meant and clarified the students´ explanations to the rest of the class in order to facilitate the exchange of students´ thinking and to foster the discussion.

4. Discussion and Conclusions

Confirming Carpenter, Franke and Levi (2003) and Koehler (2004)'s results, we have shown that third grade students are able to develop and use relational thinking for solving number sentences. The consideration of number sentences specially designed to elicit the use of this thinking as well as encouraging the use of diverse approaches to solving number sentences were two of the key elements in our in-class intervention. The true/false sentences served to help students to consider the sentence as a whole and to break their computational mindset. Discussions were also critical in promoting students' development of relational thinking. Discussions forced students to evaluate their thinking and that of others and encouraged them to organize and consolidate their mathematical thinking because they had to communicate it.

The multiple verbalizations given by the students showed their understanding of important arithmetic properties as well as their knowledge about the structure of the operations. The students pointed out fundamental mathematical ideas which are not usually made explicit in class. In this way, arithmetic teaching became less computational and students started working on generalization in an informal way which in later grades can lead to its formalization and the introduction of algebra language, as Carpenter, Franke and Levi's (2003) and Alcalá (2000) suggest. Relational thinking allows addressing two of Martin Kindt's considerations: to pay more attention to reasoning and generalization and to ease the transition to the formal study of Algebra.

A question which we did not address in this study is if the students would apply this thinking when computing in other contexts. However, considering Koehler's (2004) study, we can hypothesize that if students are more frequently encouraged to pay attention to patterns and relations, they will start to use these relational strategies for addressing their computations in other mathematical activities different from solving number sentences.

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