IN THE TRANSITION FROM ARITHMETIC TO ALGEBRA: MISCONCEPTIONS OF THE EQUAL SIGN.

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This paper describes an ongoing study where we analyze elementary students' misconceptions of the equal sign. Considering the proposal of many researchers of fostering algebraic thinking in arithmetic settings, numeric open sentences were proposed to 3rd and 5th/6th grade students in order to analyze their understanding of the equal sign and their ways of thinking of and solving equal sign arithmetic expressions. Different misconceptions and solving approaches were detected in both groups.

In this report, firstly we briefly present the early algebra proposal focusing our attention in the role of arithmetic. Secondly, we report several research studies which support and are related to our study. Finally, we show the methodology, results and conclusions of the study.

EARLY ALGEBRA AND ARITHMETIC

Different research studies as well as the experience in teaching mathematics show multiple difficulties students encounter in learning algebra, when the complexity and cognitive demand of school mathematics seem suddenly increase, causing the students to memorize rules without meaning or sense and to lose interest in mathematics. In order to overcome these difficulties and to ease the transition to algebra, many researchers propose an earlier introduction of algebra. This proposal does not mean to introduce earlier the algebra of curriculum, but changing the classroom practice in order to promote algebraic thinking previously to the high school algebra courses.

Fostering algebraic thinking from the first years of schooling can facilitate the access of all students to algebra and develop a more meaningful understanding.

Algebra has traditionally been introduced when it was considered that students have acquired the necessary arithmetic skills. Besides, it has usually been developed separately from arithmetic without taking advantage of their strong link. Nowadays, many researchers suggest working with activities which ease the transition from arithmetic to algebra (Carraher, Schliemann & Brizuela, 2000; Kaput, 2000). They claimed that the separation between algebra and arithmetic accentuate and prolong the students' difficulties and recommend integrating both in the curriculum as early as possible. Carpenter, Franke and Levi (2003), Carraher, Schliemann and Brizuela (2000) and Kaput and Blanton (2000, 2002) have already illustrated real classroom activities and discussions at elementary grades which help to develop students' algebraic thinking.

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EQUAL SIGN MISCONCEPTIONS

To understand that the equal sign expresses a relation is one of the keys when developing mathematical and particularly algebraic thinking. Students at elementary grades commonly encounter the equal sign in arithmetic activities and get use to utilize it. However, according to several research studies (Saenz-Ludlow and Walgamuth, 1998; Behr, Erlwanger and Nichols, 1980; Falkner, Karen, Levi, Linda and Carpenter, 1999), students from grades 1st to 6th show serious misconceptions about the meaning of the equal sign. These misconceptions are unnoticed while students perform arithmetical activities in which the relation between both sides of the equal sign does not need to be considered, and only a number appears on the right side of the equal sign. Nevertheless, when students encounter other kinds of sentences and especially when they work with algebraic identities they manifest serious difficulties regarding the use of the equal sign. They try to apply their rules and understanding of the equal sign: sometimes they calculate answers and others they look for relations, sometimes they consider both sides of the identity and others just one.

Many of the difficulties elementary students face when dealing with equal sign expressions show the strong influence previous and daily computation problems can exert on students. As Carraher, Schliemann and Brizuela (2000) suggest students' previous mathematics instruction seem to be one of the main causes of many of the students' difficulties in learning algebra.

The identities which students are used to encounter at school usually present the operation on the left side of the equal sign and the answer on the right side. The tendency of using this structure in all arithmetic sentences leads to serious misconceptions in the meaning of the equal sign which, according to Behr, Erlwanger and Nichols (1980) and Carpenter, Franke and Levi (2003), do not seem to change as the students get older. In Behr, Erlwanger and Nichols's study with children from six to twelve years old, they observed that the students not only seem to perceive the equal sign as a stimulus for an answer but also have definitive ideas about how identities should be written. For example, students changed the sentence 3 + 2 = 2 +3 to 3 + 2 + 2 + 3 = 10 or 3 = 3 to 3 + 0 = 3 or 3 - 3 = 0. Another students' reaction reported by Carpenter, Franke and Levi, from its study with thirty typical elementary-grade classes, is to change sentences like 8 + 4 □ = + 5 to 8 + 4= 12 +5 =17. Besides, when solving arithmetic open sentences, these students presented the following misinterpretations:

- Interpretation of the equal sign as a command to produce and answer. Students tended to interpret the equal sign as a command to produce an answer, assuming that the operation had to be on the left side of the equal sign and the answer immediately afterwards on the right side. For example: some students answered 12 to the sentence 5 + □ = 4 + 8.

- Operating all the numbers together.

  For example, 17 was an answer to 8 + 4 This response was .5 + □ =sometimes given when the students notice or were warned of the presence of the number 5.

OUR STUDY

Motivated by the aforementioned studies, the aim of our study was to detect elementary students' misconceptions regarding the meaning of the equal sign. We wanted not only to analyze students' understanding of the equal sign but also to identify the way students solved the sentences paying special attention to see if they established relations between the terms of the identities.

We considered a group of 15 3^{rd} grade students and a group of 26 5^{th} and 6^{th} grade students. We were interested in the study of these two groups because the different levels could show us significant differences of understanding: the 3^{rd} grade students would present more difficulties while the $5^{th}/6^{th}$ grade group could have acquired a suitable understanding of the equal sign. Moreover, the consideration of these two groups could let analyze differences in approaches for solving equal sign arithmetic sentences when having different conceptions.

According to the NCTM Standards (page 159), in these grades, from 3^{rd} to 5^{th}, symbolism should be introduce for representing unknown quantities and students should start to use equations for expressing mathematical relationships. A suitable understanding of the equal sign is required.

The students were proposed a series of six numeric open (it means, with an unknown quantity) sentences. We distinguished between non-action sentences and action sentences in a similar way as Behr, Erlwanger and Nichols (1980) do. We refer as non-action sentences to those with no operational sign or with at least one operational sign in each side of the equal sign, e.g. 14 + □= 13 + 4; while action sentences are those with no operational sign in just one side of the identity, e.g. □ = 25 – 12.

The students were proposed one action sentence and five non-action sentences. The only action sentence (□ = a – b) included in the series of identities was aimed to check if the student manifested problems when encountering the answer on the left side instead of the right side as they are more used to. Regarding the non-action sentences, they were elaborated varying the position of the unknown quantity and considering only four terms and addition: □ + b = c + d, a + b = □ + d, a + □= b + c, a + b = b + □. One subtraction non-action sentence was also included (a – b = □ – d). These non-action sentences were constructed in a way that allowed easily finding the unknown quantity by using relational thinking: the difference between two of the numbers in both sides was just one unit, letting deduce that an inverse relation should occurs between the other two numbers. This could facilitate the solution of the identities if the students thought about the relations between both sides of the equal sign. Concretely, one of the identities corresponded to the expression of the commutative property of addition in the case of two particular numbers.

Because these sentences were aimed to infer the students' understanding of the equal sign any extra difficulty was tried to avoid, so, only subtraction and addition within natural numbers lower than 40 were included and no more than four numbers appeared in each sentence.

RESULTS AND DISCUSSION

Analyzing the answers of the students, the aforementioned misinterpretations of the equal sign appeared: "Interpretation of the equal sign as a command to produce and answer" and "Operating all the numbers together". In addition, two different misinterpretations were observed in our study:

- Writing the same number in the closer positions in both sides of the equal sign.

For example in the sentence 14 + □ = 13 + 4, 13 was one of the answers.

- Answer to the other side's operation (when the unknown quantity was in the first or fourth place).

In some cases when the unknown quantity was in the first place, the answer to the operation on the right side was written as the unknown number in the other side. For example, for □ + 4 = 5 + 7 one of the answers was 12. Similarly, sometimes when the unknown quantity was in the fourth place, the answer to the left side operation was placed in the box. For example 19 was an answer 12 + 7 = 7 + □ .This can be interpreted as an adaptation of the (mis)conception of the equal sign as a command to produce an answer.

We also distinguished two different manifestations of the misconception "Interpretation of the equal sign as a command to produce and answer" that we refer as *command from left to right* and *command from right to left*. The first case is when it was assumed by the students that the operation had to be on the left side of the equal sign and the answer immediately afterwards on the right side. On other hand, *command from right to left* is when the unknown quantity was on the left side next to the equal sign (second position) and the students wrote the answer to the right side's operation on the left side of the equal sign. For example: 8 + 4 = 5 + □ answering 12 is result of the misconception that we called *command from left to right*, and answering 17 to 14 + □ = 13 + 4 is a case of *command from right to left*.

3rd grade students

As can be observed in table 1, the most common misconception of the equal sign was as a command to produce an answer, and concretely from left to right. For example, a student answered 1 to the sentence □+ 4 = 5 + 7 verbalizing that he got a clue by the answer, pointing to the 5. This misconception was frequently applied in the cases where the structure of the sentence facilitated it (8 + 4 = □ + 5, 13 - □ = 7 – 6, □+ 4 = 5 + 7) in other cases the responses of the students were more varied. For example, this occurs in the sentence 14 + □ = 13 + 4 were a -1 was needed in order to think of the equal sign as a command from left to right. The answers to this sequence were: 1, 3, 17, 0,-1, 7, and 13.

The students considered the equal sign as an order to perform an operation and tried to adapt this misconception to each of the sentences for producing an answer. They did not recognize in any of the sentences the necessity of equivalence between both sides of the equal sign.

Sentence	**Correct responses**	**Misconceptions (and occurrence)**
□= b – c	3	Same number in the closer positions in both sides or command from left to right (1)
b = c+ d + □	0	Command from left to right (11) Answer to the other side's operation (1)
a + □ = c + d	3	Command from right to left (2) Command from left to right (1) Same number in the closer positions in both sides (1)
a + b = □ + d	1	Command from left to right (14)
a + b = b+ □	3	Operating all the numbers together (3) Answer to the other side's operation (1)
a – b = □ – d	0	Command from left to right (13) Same number in the closer positions in both sides (1)

Table 1: 3rd grade group's misconceptions of the equal sign

In some cases, the students recurred to look to the identity from right to left, contrary to what they are used to, as a "reasonable" way to get an answer according to their unsuitable understanding of the equal sign. However, they did not easily consider this option as can be deduce from the numerous solutions given to the action sentence (The answers to □= 25 – 12 were 13, 7, 25, 20, 18, 35, and 17).

By the kind of mistakes found and the multiple answers to the sentence of structure a + b = b + □, it can also be deduced that students did not consider the relation between the terms in both sides of the equal sign. One of the students reported to have tried to solve them by "try and check".

Many different responses were also given to the sentence 14 + □ = 13 + 4. Three students gave the correct answer 3, however, one of these students commented that the numbers had been moved around, what suggested he was looking to the digits separately. This answer, unrelated to the meaning of the equal sign, is another indicator of the many difficulties students found in these sentences and can be related to some of the (mis)uses given to the equal sign as shorthand. For example the equal sign is sometimes uses to show equality between two pictures. As Carpenter, Franke and Levi (2003) recommend it should be avoided the use of the equal sign in cases where it does not express a relation between numbers, in order to prevent misconceptions.

5th and 6th grade students

The students manifested a suitable understanding of the meaning of the equal sign. The results were completely different to the 3rd grade group, not only a higher percent of correct responses was obtained but also the nature of the wrong responses was different.

Sentence	Correct answers	Nature of the wrong responses (and occurrence)
$\square = b - c$	23	Random (3)
$b = c + d + \square$	25	Random (1)
$a + \square = c + d$	24	Random (2)
$a + b = \square + d$	21	Wrong thinking about relations (1) Answer to the other side's operation (1) No answer (1) Random (2)
$a + b = b + \square$	26	
$a - b = \square - d$	15	Wrong thinking about relations (6) Operating all the numbers together (2) Command from left to right (1) No answer (1) Random (1)

Table 2: 5th and 6th grade students' responses

As it can be observed in table 2, some of the previously referred misconceptions of the equal sign appeared: "Operating all the numbers together", "Answer to the other side's operation" and "Command from left to right", however, in this case their occurrence was not significant.

The action sentence did not cause special difficulties. Regarding the non-action sentences and considering the wrong answers, it can be inferred that some students noticed and used the relations existing between the terms of the identity. It can be deduced because 7 of the 19 wrong answers are result of a wrong thinking regarding the relations. For example, in the sentence 15 – 9 = □ – 8 the three wrong answers were 16 and in 16 + 7 = □ + 8 one of the two wrong answers was 17. Besides, there are several random mistakes which could be consequence of a wrong relational.

The non-action sentence of structure a – b = □ – d caused significant difficulties to the students. In this sentence only 15 of the responses were correct contrasting with the high percent of correct responses given to the other sentences. The special difficulty which caused this sentence is also manifested by the fact that six of the nine students with only one incorrect answer had difficulties in this sentence. This fact could be consequence of the more complexity of applying relational thinking in subtraction sentences.

This table on the right side shows the distribution of the wrong responses among the student's answers.

Number of wrong responses	Number of students
0	12
1	9
2	1
3	1
4	2
5	0
6	0

CONCLUSIONS

The results obtained in the study regarding the 3rd grade group, show important difficulties students encounter when dealing with the equal sign in spite of its frequent use in arithmetic activities. The unidirectional use which students commonly see during their arithmetic learning, leads to erroneous conceptions of the equal sign. These misconceptions can cause serious problems in algebra and limits students' capacity to reflex on identities.

In our study, significant differences are observed between the 3rd and the 5th/6th grade students. Although in other research studies 5th and 6th grade students have shown serious misconceptions of the equals sign, in this case the difficulties with the use of the equal sign seemed to have been overcome by the 5th/6th grade students. In addition, when the students had a suitable understanding of the equal sign, more developed strategies for solving the sequences were detected. The students established relations between the terms of the identities. However, they presented difficulties in applying their relational thinking in subtraction contexts.

Due to the importance of a correct understanding of the equal sign, a special attention should be paid to notice the conceptions which students are acquiring. From the beginning of the equal sign's used, a wide and complete variety of arithmetic identities should be presented to elementary grade students in order to help them to develop a suitable understanding of the equal sign. It is essential to guarantee a correct understanding of arithmetic expressions before beginning the learning of algebra. A meaningful introduction of symbolism can help to avoid later superficial strategies (Fagnant, 2002).

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