THIRD GRADERS´ STRATEGIES AND USE OF RELATIONAL THINKING WHEN SOLVING NUMBER SENTENCES

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Relational thinking is an important element of algebraic thinking which has potential for promoting the integration of arithmetic and algebra in the elementary curriculum and the development of a meaningful learning of arithmetic. Focusing on the context of number sentences, we have analysed the use of relational thinking by a group of third graders. In this paper we describe the various strategies identified in the students´ production. The results evidence a great variability in the way of using relational thinking and a variable role of computation in that use.

BACKGROUND: EARLY ALGEBRA AND RELATIONAL THINKING

In the last two decades numerous researchers have analysed and promoted the integration of algebra in the elementary curriculum. This curricular proposal raises the introduction of algebraic ways of thinking in school mathematics from the first school years, aiming to foment mathematics learning with understanding and, more specifically, to ease the learning of algebra. Algebraic ways of thinking can naturally emerge from elementary mathematics and favour the students´ conceptual development of deeper and more complex mathematics, from very early ages (Blanton & Kaput, 2005). In addition, the late introduction of this type of thinking in the school curriculum is thought to be responsible, at least in part, for pupils' subsequent difficulties (Bastable & Schifter, 2007; Carraher & Schliemann, 2007).

From this view, algebra is conceptualized quite broadly, including: the study and generalization of patterns and numeric relations, the study of structures abstracted from computation and relations, the study of functional relations, the development and manipulation of symbolism, and modelling (Kaput, 1998).

Interested in analysing the transition between arithmetic and algebra as well as promoting the integration of both sub-areas, various researchers (Carpenter, Franke, & Levi, 2003; Koehler, 2004; Molina, Castro, & Ambrose, 2006; Stephens, 2007) have focused their attention on the use of relational thinking. When working with arithmetic and algebraic expressions, relational thinking imply to consider expressions as a whole, analysing them to find their inner structure, and exploiting these relations to construct a solution strategy. It is equivalent to what Henjny, Jirotkova, & Kratochvilova (2006) call "conceptual meta-strategies".

Relational thinking has been mainly considered in the context of number sentences, where it has also been referred as analysing expressions (Molina & Ambrose, 2008). In this context, its use avoids computing the numeric value of each side of the sentence. For example, when considering the number sentence 5 + 11 = 6 + □ some students may notice that both expressions include addition and that one of the addends on the left side, 5, is one less than the addend on the other side, 6. Noticing this relation and having an (implicit or explicit) understanding of addition properties, enable students to solve this problem without having to perform the computations 5 plus 11 and 16 minus 6.

Similarly when solving equations such as $\frac{1}{4}-\frac{x}{x-1}-x=5+\left(\frac{1}{4}-\frac{x}{x-1}\right)$, instead of operating on the variables and the numbers and regrouping them, students may pay attention to the structure and appreciate that this equation is equivalent to - x = 5 as the expression $\frac{1}{4}-\frac{x}{x-1}$ is repeated in both sides (Hoch and Dreyfus, 2004).

This type of thinking implies the use of number sense and operation sense (as defined by Slavit, 1999) as well as structure sense (Linchevski & Livneh 1999; Hoch & Dreyfus, 2004). It promotes a structural learning of arithmetic by leading the attention to the structure of the expressions; in this way it contributes to the development of a good base for the formal study of algebra.

Previous studies (Carpenter et al., 2003; Koehler, 2004; Molina & Ambrose, 2008) have provided evidence that elementary students are capable of using this type of thinking when solving number sentences, overcoming some issues such as the “lack of closure” and an operational understanding of the equal sign. Even when it is not addressed in teaching, students follow a linear progression in the use of relational thinking as result of their arithmetic experience (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Stephens, 2007). The extent of the acquisition of this type of thinking is varied and there are students who seem unable to use it.

Although students predominantly tend to use computational strategies, some naturally and spontaneously use relational thinking, and when teaching is designed so that this use is promoted, many students apply this type of thinking for solving some number sentences (Molina & Ambrose, 2008).

Considering these evidences, some unexplored questions which are open to research are: when and how do students´ evidence use of relational thinking, what conditions this use, which differences are there between different students´ use, how students´ develop and progress in their use. Our aim in this paper is to provide partial responses to some of these questions by describing the different ways in which a group of third grade students applied relational thinking along a teaching experiment.

DESIGN OF THE STUDY

Our research method shared the features of design experiments identified by Cobb and his colleagues (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) (See Molina, Castro, & Castro, 2007, for further details). We have developed a teaching experiment in which we worked with a group of 26 eight-year old Spanish students during six sessions over a period of one year. In this paper we will mainly focus on the data gathered on the last four sessions as the first two were directed to exploring and extending students' understanding of the equal sign. The general aim of this research work was to study students' thinking involved in solving number sentences, in the context of whole class activities and discussions. We analysed the strategies that students used to solve the sentences, focusing on detecting evidences of use of relational thinking.

The tasks used were number sentences, mostly true/false number sentences (e.g. $72 = 56 - 14$, $7 + 7 + 9 = 14 + 9$, $10 + 4 = 4 + 10$) which were proposed to the students in written activities, in whole-class discussions and in interviews. All the sentences used were based on some arithmetic property or principle (e.g., commutative property, inverse relation of addition and subtraction, compensation relation) and, therefore, could be solved by using relational thinking.

We did not promote the learning of specific relational strategies but the development of a habit of looking for relations, trying to help students to make explicit and apply the knowledge of structural properties which they had from their previous arithmetic experience. Students' use of relational thinking was favoured by encouraging them to look for different ways of solving the same sentence and showing a special appreciation of students´ explanations based on relations.

STUDENTS´ STRATEGIES

We describe here the strategies identified in the students´ responses to the proposed number sentences[1]. They differ in the role of computation as well as in the moment of the solving process and the way in which students used relational thinking. We distinguish two group of strategies depending on the motivation that initially guided the strategy: (a) to make calculations to **F**ind and **C**ompare the numerical values of both sides (type FC), or (b) to **L**ook at the sentence and to **D**etect particular characteristics of it or relations between its elements (type LD).

Within the first group, we distinguish two different approaches. Sometimes students followed their initial tendency and solved the sentence by comparing the numerical values obtained (strategy type O, operational). Others, however, in the process of performing the calculation, students changed their strategy after appreciating some characteristic of the sentence or some relations between its terms, not previously noticed (strategy type IC, interruption of computation). This observation leaded them to solve the sentence without finishing the calculation. In these cases, initiating the calculation process served the student to become aware of the structure and elements of the sentence.

Within each of these approaches, we distinguish various strategies (see Figure 1) which we describe below. The strategies identified describe different ways in which relational thinking was used by the students to solve the proposed number sentences.

[1] Our identification of the strategy used by the students is based on their production, i.e., on their thinking made explicit through their explanations. It is not always possible to exactly determine which strategy was used in each case; an answer can sometimes be result of more than one strategy. This fact is mostly due to the briefness of some answers or to the occasional lost of temporality of the actions expressed by the students.

We observe that, although in some strategies type O there is not use of this type of thinking, in all the other strategies it is evidenced in some way. The sophistication of its use and the influence in it of the computation process is variable.

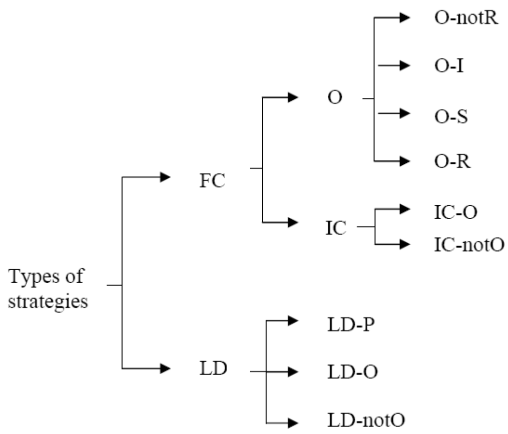

Figure 1. Classification of students´ strategies.

Strategies type O

In all strategies type O, students perform operations involved in the sentence and conclude the answer by comparing the numeric values obtained. Students display some dependence on the computation of the numeric values of each side in order to determine if the sentence is true or false. Within strategies type O, we distinguish four special ones: O-notR, O-I, O-S and O-R.

O-notR consists of performing the operations in both sides of the sentence without appreciating any relation or special characteristic of the sentence. In the strategy O-I, before doing any computation, students pay attention to the terms or operations in the sentence, in order to decide about the best way to approach the computation. This strategy evidences the simplest way in which the use of relational thinking was displayed. The following students´ explanations suggest the use of these two strategies respectively:

"[In the sentence $17-12=16-11$] true because I did one operation and the other and I got the same" (She computes 17 - 12 = 05 y 16 - 11 = 05 by the standard algorithm).

"[In the sentence $19-3=18-2$] Cause two is less than three… [Researcher: Yes]... so, it is easier to subtract it, then I have subtracted eighteen minus two, which is easier, and I got sixteen, and the other nineteen minus three is sixteen".

In the strategy O-S (sameness), the use of relational thinking is also fairly basic. O-S consists on the appreciation of sameness between the operations to perform in one side and those already performed in the other side, which allows avoiding some computation. This strategy evidences that part of the student's attention is not focus on the computation, allowing him or her to distinguish sameness between the terms which is operating. For example, we identify a possible use of this strategy in the following explanation given to the sentence $7+7+9=14+9$: "I added seven plus seven, which gives fourteen, and plus nine is twenty-three. And, then, I have seen the fourteen plus nine and they are twenty-three". In this case we need to pay attention to the student's tone voice for noticing that she identified the operation on the right side as one of the operations previously performed when computing the numeric value of the left side.

In the strategy O-R, students detect some relations or characteristics of the sentence while or after computing the numeric values of both sides. This observation allows them to determine if the sentence is true or false by using some related arithmetic knowledge (not just number facts). When using this strategy, students provide two justifications of their answer: one based on the comparison of the numeric values of both sides and other based on the appreciated relations or characteristics. For example, we identify a possible use of this strategy in the following explanation given by a student to the sentence $7+15=8+15$: "False, because it doesn't give the same, and because seven is smaller" (Aside she computes the numeric value of both sides by using the addition standard algorithm). This student appreciates a difference of magnitude between the numbers contained in both sides of the sentence.

Here the use of relational thinking is more sophisticated than the ones previously mentioned as students consider the sentence as a whole, make distinctions and appreciate relations between its terms. This use is dependent on the performance of the computation which helps the students to become aware of the components of the sentence and to relate them. This strategy also evidences some reliance on the comparison of the numeric values of both sides in order to decide and justify if the sentence is true or false.

Strategies type IC

Like in the strategy O-R, strategies type IC evidence a use of relational thinking connected to performing some computations, as students appreciate relations or particular characteristics of the sentence through performing some computations. Students may be influenced by some tendency to operate which leads them to start computing before looking to the sentence or they may require computing for becoming aware of the structure of the sentence and the elements that it contains. However, in this case students abandon the computation initiated and don't need it to determine and justify if the sentence is true or false.

We distinguish two strategies type IC, IC-O and IC-notO, depending on if the appreciated relations or characteristics lead the student to know the numeric value of both sides or not. The following students´ explanations suggest the use of these two strategies respectively.

[In the sentence $257-34=257-30-4$] "False because instead of subtracting 34 they subtract 30 and the four goes aside" (Aside she uses the subtraction standard algorithm to calculate $257-34=1223$, $257-30=227$ and $227-4=223$).

[In the sentence $51+51=50+52$] “Because fifty-one plus fifty-one is one hundreds and two, but fifty-one, if you subtract [one], fifty, you can add [it] to the other fifty-one, one more, and you get fifty-two”.

Strategies type LD

As previously explained, in strategies type LD students tackle the resolution of the sentence by observing it and looking for relations or especial characteristic of it. They display an initial disposition to use relational thinking and don’t show any dependence on performing computations. So, we consider these are the strategies which display the most sophisticated use of relational thinking.

We distinguish between three strategies type LD. LD-O and LD-notO differ on if the appreciated relations or characteristics lead the student to know the numeric value of both sides or not. The following students´ explanations suggest the use of these two strategies respectively:

[In the sentence $125-125=13$] “False, because you subtract one hundred and twenty-five to one hundreds and twenty-five and it is zero, not thirteen [Researcher: How do you know that it is zero?] Because here there are the same numbers, and if you subtract the same numbers it is zero, here it cannot be thirteen”. He applies the property $a-a=0$ after observing the sameness of the terms in the left side.

[In the sentence $75+23=23+75$] “True because in addition the order doesn’t matter”. She notices the sameness of terms in both sides, although in different order, and use the commutative property to conclude the trueness of the sentence without needing the numeric value of both sides.

The other strategy, called LD-P (prediction) consists of the use of two strategies: (1) LD-O or LD-notO, to determine if the sentence is true or false, and (2) a strategy type FC to justify the answer to the sentence. We detect evidences of the use of this strategy in the interview to a student about the sentence $11-6=10-5$. Initially she concluded that the sentence was true after thinking for some seconds, but when we asked her why, she started computing the numeric values of both sides and said: “Because if you subtract six from eleven is…five, and if you subtract five to ten, five”. When being asked if she could explain it in other way, she referred to differences in magnitude between the terms in both sides: "Because… if eleven is higher than ten and you subtract one more than five, you get the same”. The way in which she provided this explanation, without taking time to think, suggests that she had previously appreciated this relationship.

Discussion and Conclusions

We have described the strategies used, along the teaching experiment, by a group of third graders when solving number sentences based on arithmetic properties. We focused the analysis of these strategies on the use of relational thinking evidenced. Within the results, we want to highlight the different grades of sophistication detected in this use. The most basic ones are (a) paying attention to the structure and composition of the sentence to decide about the best way to approach computations, as well as (b) appreciating sameness between the operations being done while computing. This use does not require recalling any special arithmetical property or principle. The most sophisticated use of relational thinking does not involve computing but considering the sentence as a whole, recognizing its structure and appreciating relations between its terms which allow solving the sentence.

Computation was identified as an important element for becoming aware of the composition of the sentence. Through the action of making computations, some elements or relations between elements "stood out to the students´ eyes". Their attention got caught by some particularities of the sentence and then some related arithmetic knowledge got into play. Probably this dependence on (or tendency to) operating is consequence of the strong computational approach of traditional arithmetic teaching.

These results enrich our knowledge about elementary students´ use of algebraic ways of thinking in arithmetic context, which is key for the integration of both sub areas in the elementary curriculum. Although "thinking relationally while computing" is strongly valuable and desirable, in order to get students to use relational thinking in algebraic context, "thinking relationally without computing" need also to be promoted (as computation is not always possible when working with algebraic expressions).

Endnote

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