GEOMETRIC CONSTRUCTIONS ON SPHERES AND PLANES IN $\mathbb{R}^n$

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Abstract

Using Lie geometry and the Lie product in $\mathbb{R}^{n+3}$, we give an algebraic description of geometric objects constructed from spheres and planes of dimension $n-k$, $k \geq 1$ in $\mathbb{R}^n$. We define algebraic invariants, which characterize geometric properties of these objects, and their position in $\mathbb{R}^n$.

1. Introduction

In Lie geometry, oriented spheres and planes of dimension $n-1$ in $\mathbb{R}^n$ are described as points on a quadric surface, called the Lie quadric, in the projective space $\mathbb{P}^{n+2}$. Geometric relations like tangency, angle of intersection, power, etc., are described by the Lie product, which is an indefinite bilinear form on the space $\mathbb{R}^{n+3}$. Lie geometry enables writing geometric relations in terms of algebraic equations, and is thus an appropriate environment for automatic solving of geometric constructions and for proving certain geometric theorems. Lie geometry has been used to study geometric problems on circles for example in [8], [7], [3]. A thorough treatment of Lie geometry can be found in [1] or [2].

A point, an oriented plane of dimension $n-1$, or an oriented sphere of dimension $n-1$ in $\mathbb{R}^n$ is called an *oriented geometric cycle*. In [4], the existence and properties of solutions of certain types of geometric constructions on oriented geometric cycles in $\mathbb{R}^n$ were analyzed. In [5], simple algorithms for symbolic solutions of a number of such geometric constructions using the Apollonius construction and Lie transformations were given. In this paper, we attempt to generalize this approach to other geometric objects. For example, spheres and planes of codimension 2 in $\mathbb{R}^n$, which we call *geometric subcycles* are described triples of cycles, where the last element of the triple is the special cycle r. On the other hand, a triple of cycles with the last element equal to the dual special cycle w, determines a cone in $\mathbb{R}^n$. Both of these constructions - subcycles and cones are specific examples of a general construction, which is given by a $k+1$-tuple of cycles, where the first k-cycles are proper, and the last component is a special cycle which determines the geometric nature of the object. Certain geometric properties of such objects are determined by the signature of the Lie from restricted to the subspace spanned by homogeneous coordinates of these cycles, expressed in appropriate local coordinates.

2. Cycles

We will call an element $x \in \mathbb{P}^{n+2}$ (denoted by a lower case letter) an *algebraic cycle* (or, mostly, just a cycle). An algebraic cycle x is given by a nonzero vector of homogeneous coordinates $X = (X_0, \dots, X_{n+2}) \in \mathbb{R}^{n+3}$ which we denote by the corresponding capital letter.

The *Lie product* on $\mathbb{R}^{n+3}$ is the indefinite bilinear form with signature $(n-1, 2)$ given by

$$(X \mid Y) = X_0 Y_{n+1} + X_1 Y_1 + \cdots + X_n Y_n + X_{n+1} Y_0 - X_{n+2} Y_{n+2} = X^T \mathbf{A} Y, \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{0} & 1 & 0 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & 0 \\ 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2}$$

The set of vectors X such that $(X \mid X) = 0$ determines the *Lie quadric*

$$\Omega = \{x \in \mathbb{P}^{n+2} \mid (X \mid X) = 0\} \subset \mathbb{P}^{n+2}.$$

Cycles $x \in \Omega$ will be called proper cycles, while cycles $x \notin \Omega$ will be called non-proper cycles.

If $X_1, \dots, X_k$ are homogeneous coordinate vectors, the symbol $\langle X_1, \dots, X_k \rangle$ will stand for the linear subspace spanned by the vectors $X_1, \dots, X_k \in \mathbb{R}^{n+3}$, and the symbol $\langle X_1, \dots, X_k \rangle^\perp$ for the orthogonal complement to $\langle X_1, \dots, X_k \rangle$ with respect to the Lie product, i.e.

$$\langle X_1, \dots, X_l \rangle^\perp = \{Y \mid (X_i \mid Y) = 0, i = 1, \dots, k\}.$$

Following our convention on capital and lowercase letters, $\langle x_1, \dots, x_k \rangle$ and $\langle x_1, \dots, x_k \rangle^\perp$ will denote the projective subspace spanned by $x_1, \dots, x_k$ and its dual projective subspace respectively.

For any vector $S \in \mathbb{R}^{n+3}$, the open set $\mathcal{U}_s = \mathbb{P}^{n+2} \setminus \langle s \rangle^\perp$ together with the map

$$\varphi_S \colon \mathcal{U}_s \cong \langle S \rangle^\perp \cong \mathbb{R}^{n+2}, \quad \varphi_S(x) = \frac{1}{(X \mid S)} X$$

is a chart on $\mathbb{P}^{n+2}$. The collection

$$\{(\mathcal{U}_s, \varphi_S) \mid S \in \S^{n+2}\}$$

determines an atlas which gives the standard manifold structure on $\mathbb{P}^{n+2}$. Two cycles and their corresponding charts have a special role in Lie sphere geometry: the nonproper cycle r with homogeneous coordinates $R = (0, \dots, 0, 1)$ and the proper cycle w with homogeneous coordinates $W = (1, 0, \dots, 0)$ which we call the *infinite cycle*.

An oriented geometric cycle, i.e. an oriented sphere or plane of codimension 1 or a point, in $\mathbb{R}^n$ is represented by a proper algebraic cycle $x \in \Omega \subset \mathbb{P}^{n+2}$ in the following way.

- A point $\mathbf{p} \in \mathbb{R}^n$ is represented by the cycle $x \in \mathcal{U}_w$ with local coordinates $\varphi_W(x) = (-\|p\|^2/2, \mathbf{p}, 1, 0)$.

- The positively oriented (i.e. outward normal) and negatively oriented sphere with center $\mathbf{p}$ and radius $|\rho| > 0$ can be represented by the cycles x and x' in $\mathcal{U}_w \cap \mathcal{U}_r$ with local coordinates

$$\varphi_W(x) = \left(\frac{\rho^2 - \|\mathbf{p}\|^2}{2}, \mathbf{p}, 1, \rho\right), \quad \varphi_R(x) = \left(\frac{\rho^2 - \|\mathbf{p}\|^2}{2\rho}, \frac{\mathbf{p}}{\rho}, \frac{1}{\rho}, 1\right)$$

$$\varphi_W(x') = \left(\frac{\rho^2 - \|\mathbf{p}\|^2}{2}, \mathbf{p}, 1, -\rho\right), \quad \varphi_R(x') = \left(\frac{\|\mathbf{p}\|^2 - \rho^2}{2\rho}, -\frac{\mathbf{p}}{\rho}, -\frac{1}{\rho}, 1\right)$$

  respectively.

- The plane with normal $\mathbf{n}$, where $\|\mathbf{n}\| = 1$, and point $\mathbf{q}$ is represented by the cycle $x \in U_r$ with local coordinates

$$\varphi_R(x) = (-\mathbf{n} \cdot \mathbf{q}, \mathbf{n}, 0, 1).$$

On the other hand every proper cycle x, except the infinite cycle w represents an oriented geometric cycle in $\mathbb{R}^n$. It follows from the equation $(X \mid X) = 0$ that $\Omega \subset \mathcal{U}_w \cup \mathcal{U}_r$. The complement

$$\Omega \setminus \mathcal{U}_w = \{x \in \Omega \mid (X \mid W) = 0\} \subset \mathcal{U}_r$$

consists of cycles representing planes and the infinite cycle w, while the complement

$$\Omega \setminus \mathcal{U}_r = \{x \in \Omega \mid (X \mid R) = 0\} \subset \mathcal{U}_w,$$

consists of cycles representing points in $\mathbb{R}^n$ and the infinite cycle w. A change of sign of the last homogeneous coordinate of a cycle $x \in \Omega$, produces the *reoriented cycle* $x' \in \Omega$ representing the same nonoriented geometric cycle with the opposite orientation. Points have only one possible orientation. If x is a point, then $x' = x$.

Remark 1. *Spheres and planes in $\mathbb{R}^n$ correspond through the stereographic projection to spheres in S^{n+1}, and in this setting, the cycle w is the representation of the pole in S^{n+1}.*

Motivated by the geometric background we will use the following notation. The components of the vectors of homogeneous coordinates will be denoted by $X = (X_v, X_{\mathbf{p}}, X_\omega, X_\rho)$, where $X_v, X_\omega, X_\rho \in \mathbb{R}$ and $X_{\mathbf{p}} \in \mathbb{R}^n$. Thus, in the chart $\mathcal{U}_w$ where $X_\omega = 1$, the last coordinate X_ρ determines the radius $\rho = |X_\rho|$ and the orientation of the sphere associated to x and $X_{\mathbf{p}}$ is its center, and in the chart $\mathcal{U}_r$ where $X_\rho = 1$, the coordinate X_ω is the curvature of the geometric cycle associated to x.

If x is any proper cycle, then the vector of homogeneous coordinates

$$X^r = X + (X \mid R)R \in \langle R \rangle^\perp$$

represents the *Möbius coordinates* of the non-oriented geometric cycle determined by x, and the product $(X^r|X^r)$ corresponds to the Möbius product [2]. In Möbius geometry a non-oriented sphere or plane in $\mathbb{R}^n$ is represented by a point in $\mathbb{P}^{n+1}$. The projection from Ω to $\mathbb{P}^{n+1}$ corresponds to assigning to an oriented geometric cycle the underlying non-oriented one. If $x \in \Omega$ then

$$(X \mid X) = (X^r \mid X^r) - (X_\rho)^2 = 0 \quad \text{so} \quad (X^r \mid X^r) = (X_\rho)^2. \tag{3}$$

A crucial property of the Lie product on homogeneous coordinates is that it tells us when two cycles are tangent with compatible orientations.

Proposition 1. *Let x_1 and x_2 be proper cycles such that $(X_1 \mid X_2) = 0$. If one of the cycles, for example x_1, is a point cycle then it lies on x_2. If both x_1 and x_2 are non-point cycles then they are tangent with compatible orientations. If both are planes then this means that they are parallel with compatible orientations.*

The proof of this proposition amounts to simple geometric verifications, and can be found for example in [2].

We will see in the following sections that the Lie product of homogeneous coordinate vectors reflects several other geometric properties of the corresponding pair of cycles.

3. Families of cycles

As we have seen, a proper cycle $x \in \Omega \subset \mathbb{P}^{n+2}$ represents an oriented geometric cycle in $\mathbb{R}^n$. A family of proper cycles thus represents a family of oriented geometric cycles, which determine new geometric objects. For example, two intersecting geometric cycles determine a subcycle, i.e. a cycle of codimension 2 in $\mathbb{R}^n$, and two spheres can determine the common tangent cone in $\mathbb{R}^n$. On the algebraic side, a family of cycles $x_1, \ldots, x_k$ spans the projective subspace $\langle x_1, \ldots, x_n \rangle \subset \mathbb{P}^{n+2}$. This subspace can be projected onto Ω in different directions which are determined by a further cycle s. The projection $\langle x_1, \ldots, x_n, s\rangle \cap \Omega$ determines a $k-1$-parametric family of proper cycles, which is the algebraic encoding of the corresponding family of geometric cycles. For example, if x, y represent intersecting cycles and $s = r$, then $\langle x, y, r\rangle \cap \Omega$ determines the Steiner pencil spanned by x and y, i.e. the family of all geometric cycles intersecting in a common subcycle, and $\langle x, y, r\rangle^\perp \cap \Omega$ determines the family of points forming this subcycle. Similarly, if x and y are spheres with a common tangent cone, then $\langle x, y, w\rangle \cap \Omega$ determines all cycles tangent to this cone, and $\langle x, y, w\rangle^\perp \cap \Omega$ is the collection of the common tangent planes.

In the following sections we will discuss in what way algebraic properties of the subspaces $\langle x_1, \ldots, x_n, s\rangle \subset \mathbb{P}^{n+2}$ determine the geometric properties of the corresponding geometric object in $\mathbb{R}^n$. In this section we will define some such algebraic properties. For us, the most important such property is the signature of the restriction of the Lie form,

Let $X_1, \ldots, X_k \in \mathbb{R}^{n+3}$ be a k-tuple of linearly independent vectors, and let $\mathbf{X} = [X_1, \ldots, X_k]$ denote the $k \times (n+3)$ matrix with columns $X_1, \ldots, X_k$,. The Lie form, restricted to the subspace $\langle X_1, \ldots, X_k\rangle$, is given by the $k \times k$ matrix

$$\mathbf{A_X} = \mathbf{X}^T \mathbf{A} \mathbf{X}.$$

The determinant of this matrix will be denoted by

$$\det(\mathbf{A_X}) = \Delta(\mathbf{X}) = \Delta(X_1, \ldots, X_n). \tag{4}$$

The sign of $\Delta(\mathbf{X})$ depends on the signature of the Lie form on the subspace $\langle X_1, \dots, X_k \rangle$, and is an invariant of this subspace, independent of the choice of basis.

Proposition 2. *Let* $0 < k < n+2$ *and let* $\mathbf{X} = [X_1, \dots, X_k]$ *be such that* $\Delta(\mathbf{X}) < 0$. *Then*

$$\langle x_1, \dots, x_k \rangle^{\perp} \cap \Omega \neq \emptyset.$$

Demostración. Since $\Delta(\mathbf{X}) < 0$, the Lie form on $\langle X_1, \dots, X_k \rangle$ and on the complement $\langle X_1, \dots, X_k \rangle^{\perp}$ is nondegenerate, and

$$\mathbb{R}^{n+3} = \langle X_1, \dots, X_k \rangle \oplus \langle X_1, \dots, X_k \rangle^{\perp}$$

(compare [2, Theorem 1.2]). Let $Y_1, \dots, Y_{n+3-k}$ be a basis of $\langle X_1, \dots, X_k \rangle^{\perp}$, and $\mathbf{Z} = (X_1, \dots, X_k, Y_1, \dots, Y_{n+3-k})$. Then

$$\Delta(\mathbf{Z}) = \Delta(\mathbf{X})\Delta(\mathbf{Y}) > 0,$$

so $\Delta(\mathbf{Y}) < 0$, and the Lie form restricted to $\langle X_1, \dots, X_k \rangle^{\perp}$ is indefinite (in fact it has signature $(n+3-k, 1)$). But then there exists a vector $U \in \langle X_1, \dots, X_k \rangle^{\perp}$ such that $(U \mid U) = 0$, and so $u \in \Omega$. □

We will use mostly determinants in which one vector $S \in \mathbb{R}^{n+3}$ has a special role, so it is reasonable to introduce a new notation: for any k-tuple of cycles with homogeneous coordinates $X_1, \dots, X_k$, the $(k+1) \times (k+1)$ determinant

$$\Delta(X_1, \dots, X_k, S) = \det([X_1, \dots, X_k, S]^T A [X_1, \dots, X_k, S]) \tag{5}$$

will be called the S*-determinant* of the k-tuple $X_1, \dots, X_k$.

In the case $k = 1$, the S-determinant of the homogeneous coordinates X of a single proper cycle $x \in \Omega$ is

$$\Delta(X, S) = \begin{vmatrix} 0 & (S \mid X) \\ (X \mid S) & (S \mid S) \end{vmatrix} = -(X \mid S)^2. \tag{6}$$

In particular, if $x \in \mathcal{U}_s$, then $\Delta(\varphi_S(x), S) = -1$.

The two canonical choices for S will be $S = R$, and $S = W$. For example, by Proposition 2 it follows that:

Corollary 1. *Two geometric cycles, described by algebraic cycles* x, y *intersect if* $\Delta(X, Y, R) \leq 0$, *and have a common tangent plane is* $\Delta(X, Y, W) \leq 0$.

Let us compute some examples of R and W determinants.

1. The R-determinant of a single proper cycle $x \in \mathcal{U}_w$ in the local coordinates of $\mathcal{U}_w$ equals
$$\Delta(\varphi_W(x), R) = -(\varphi_W(x) \mid R)^2 = -\rho^2$$
where ρ is the radius of the sphere represented by x.

2. The R-determinant of a pair of homogeneous coordinate vectors X_1, X_2 of proper cycles $x_1, x_2 \in \Omega$ is

$$\begin{aligned}\Delta(X_1, X_2, R) &= \begin{vmatrix} 0 & (X_1 \mid X_2) & -X_{1\rho} \\ (X_1 \mid X_2) & 0 & -X_{2\rho} \\ -X_{1\rho} & -X_{2\rho} & -1 \end{vmatrix} \\ &= (X_1 \mid X_2)((X_2 \mid X_1) + 2X_{1\rho}X_{2\rho}) \\ &= ((X_1^r \mid X_2^r) - X_{1\rho}X_{2\rho})((X_1^r \mid X_2^r) + X_{1\rho}X_{2\rho}) \\ &= (X_1 \mid X_2)(X_1 \mid X_2')\end{aligned}$$

where x_2' is there oriented cycle x_2.

3. Let $x_1, x_2 \in \mathcal{U}_r \cap \Omega$ represent intersecting non-point cycles. If both of the cycles are spheres, then, by the law of cosines,

$$(\varphi_R(x_1)^r \mid \varphi_R(x_2)^r) = \frac{\|\mathbf{p}_1 - \mathbf{p}_2\|^2 - \rho_1^2 - \rho_2^2}{2\rho_1\rho_2} = \cos\psi$$

where ψ is the angle of intersection of the two spheres.

Similarly, if one of the cycles is a plane,

$$(\varphi_R(x_1)^r \mid \varphi_R(x_2)^r) = \frac{\mathbf{np} - \mathbf{q}}{\rho} = \cos\psi,$$

where $\mathbf{q}$ is a point of intersection, and if both cycles are planes

$$(\varphi_R(x_1)^r \mid \varphi_R(x_2)^r) = \mathbf{n}_1\mathbf{n}_2 = \cos\psi.$$

So in any case

$$\Delta(\varphi_R(x_1), \varphi_R(x_2), R) = (\varphi_R(x_1)^r \mid \varphi_R(x_2)^r)^2 - 1 = \cos^2\psi - 1 = -\sin^2\psi,$$

4. If $x_1, x_2 \in \mathcal{U}_w \cap \Omega$ are intersecting spheres then $\varphi_W(x_i) = \rho_i\varphi_R(x_i)$, so

$$\Delta(\varphi_W(x_1), \varphi_W(x_2), R) = -\rho_1^2\rho_2^2\sin^2\psi = -4a^2,$$

where a is the area of the triangle with vertices $\mathbf{p_1}, \mathbf{p}_2$, and $\mathbf{q}$, and $\mathbf{q}$ is a point in the intersection.

5. Let $x_1, x_2 \in \mathcal{U}_w \cap \Omega$ be proper non-plane cycles. Then

$$\Delta(\varphi_W(x_1), \varphi_W(x_2), W) = (\rho_1 - \rho_2)^2 - \|\mathbf{p}_1 - \mathbf{p}_2\|^2.$$

If this is negative, then the two geometric spheres either intersect or one lies in the exterior of the other. A common tangent plane exists and

$$\Delta(\varphi_W(x_1), \varphi_W(x_2), W) = -P^2$$

where P is the tangential distance. If it is positive, then one sphere lies in the interior of the second one, and the geometric meaning of

$$\Delta(\varphi_W(x_1), \varphi_W(x_2), W)$$

is, at least in our situation, less relevant.

6. Let $x_1, x_2, y \in \mathcal{U}_r \cap \Omega$ be proper non-point cycles, and let the geometric cycles given by x_1 and x_2 intersect orthogonally. If $X_i = \varphi_R(x_i), i = 1, 2$ and $Y = \varphi_R(y)$ then $(X_1 \mid X_2) = -1$ and

$$\begin{aligned}
\Delta(X_1, X_2, Y, R) &= \begin{vmatrix} 0 & -1 & (X_1 \mid Y) & 1 \\ -1 & 0 & (X_2 \mid Y) & 1 \\ (Y \mid X_1) & (Y \mid X_2) & 0 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix} \\
&= 1 + 2(X_1 \mid Y) + (X_1 \mid Y)^2 + 2(X_2 \mid Y) + (X_2 \mid Y)^2 \\
&= ((X_1 \mid Y) + 1)^2 + ((X_2 \mid Y) + 1)^2 - 1 \\
&= (X_1^r \mid Y^r)^2 + (X_2^r \mid Y^r)^2 - 1 \\
&= \Delta(\varphi_R(x_1), \varphi_R(x_2), \varphi_R(y), R) \\
&= \cos^2\psi_1 + \cos^2\psi_2 - 1,
\end{aligned}$$

where ψ_1 and ψ_2 are the angles of intersection between x_1 and y, and x_2 and y, respectively.

Another important algebraic tool which we will use is the Lie orthogonal projection onto the subspace $\langle x_1, \ldots, x_k, s\rangle$. Given a k-tuple of vectors $\mathbf{X} = [X_1, \ldots, X_k]$ such that $\Delta(\mathbf{X}) \neq 0$, the Lie-orthogonal projection

$$P_{\mathbf{X}} : \mathbb{R}^{n+3} \to \langle X_1, \ldots, X_k\rangle \subset \mathbb{R}^{n+3}$$

is given by

$$P_{\mathbf{X}}Y = \mathbf{X}\mathbf{A}_{\mathbf{X}}^{-1}\mathbf{X}^T\mathbf{A}Y \tag{7}$$

To see that this really is the projection we have to check that it is idempotent:

$$P_{\mathbf{X}}^2 = (\mathbf{X}\mathbf{A}_{\mathbf{X}}^{-1}\mathbf{X}^T\mathbf{A})(\mathbf{X}\mathbf{A}_{\mathbf{X}}^{-1}\mathbf{X}^T\mathbf{A}) = \mathbf{X}\mathbf{A}_{\mathbf{X}}^{-1}\mathbf{A}_{\mathbf{X}}\mathbf{A}_{\mathbf{X}}^{-1}\mathbf{X}^T\mathbf{A} = P_{\mathbf{X}},$$

and that it is the identity on the subspace $\langle X_1, \ldots, X_k\rangle$, i.e. $P_{\mathbf{X}}\mathbf{X} = \mathbf{X}$, and maps the orthogonal subspace $\langle X_1, \ldots, X_k\rangle^\perp$ to 0, which follows directly from (7).

Since $P_{\mathbf{X}}$ depends only on the subspace spanned by $\mathbf{X}$ and not on the vectors themselves, we will use the notation $P_{\mathbf{X}^\perp}$ for the projection onto the the orthogonal subspace $\langle X_1, \ldots X_k\rangle^\perp$. Clearly, $P_{\mathbf{X}^\perp} = Id - P_{\mathbf{X}}$. The projective map determined by $P_{\mathbf{X}}$ will be denoted by

$$P_{\mathbf{x}} : \mathbb{P}^{n+2} \setminus \langle x_1, \ldots, x_k\rangle^\perp \to \mathbb{P}^{n+2}.$$

If x is a single cycle, the condition $\Delta(X) \neq 0$ is reduced to $(X \mid X) \neq 0$. The projection $\langle X\rangle$ is therefore defined only for vectors representing non-proper cycles x, and is given by

$$P_X(Y) = \frac{(X \mid Y)}{(X \mid X)}X.$$

The corresponding projective map is not very interesting, since it is the constant map

$$P_x : \mathbb{P}^{n+2} \setminus \{\langle x\rangle^\perp\} \to \{x\}.$$

The dual projection $P_{x^\perp} : \mathbb{P}^{n+2} \setminus \langle x \rangle \to \langle x \rangle^\perp$ is interesting, though. It is given by

$$P_{X^\perp} Y = Y - \frac{(X \mid Y)}{(X \mid X)} X.$$

For example, the projection $P_{r^\perp}$ is given by

$$P_{R^\perp} Y = Y + \frac{(R \mid Y)}{R}.$$

If a vector y is a proper cycle, then

$$P_{R^\perp} Y = Y^r = Y + (Y \mid R)$$

is a vector of homogeneous Möbius coordinates of this cycle.
We will often use the following algebraic result.

Proposition 3. *Let* $\mathbf{X} = [X_1, \ldots, X_k]$, $\mathbf{Y} = [Y_1, \ldots, Y_m]$. *Then*

$$\Delta(X_1, \ldots, X_k, Y_1, \ldots, Y_m) = \Delta(\mathbf{Y}) \Delta(P_{\mathbf{Y}^\perp} \mathbf{X}).$$

Demostración.

$$\begin{aligned} \Delta(\mathbf{Z}) &= \begin{vmatrix} \mathbf{X}^T \mathbf{A} \mathbf{X} & \mathbf{X}^T \mathbf{A} \mathbf{Y} \\ \mathbf{Y}^T \mathbf{A} \mathbf{X} & \mathbf{Y}^T \mathbf{A} \mathbf{Y} \end{vmatrix} \\ &= \det(\mathbf{X}^T \mathbf{A} \mathbf{X}) \det(\mathbf{Y}^T \mathbf{A} \mathbf{Y} - \mathbf{Y}^T \mathbf{A} \mathbf{X} (\mathbf{X}^T \mathbf{A} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{A} \mathbf{Y}) \\ &= \Delta(\mathbf{X}) \det(\mathbf{Y}^T \mathbf{A} P_{\mathbf{X}^\perp} \mathbf{Y}) \\ &= \Delta(\mathbf{X}) \det((P_{\mathbf{X}^\perp} \mathbf{Y})^T \mathbf{A} P_{\mathbf{X}^\perp} \mathbf{Y}) \\ &= \Delta(\mathbf{X}) \Delta(P_{\mathbf{X}^\perp} \mathbf{Y}) \quad \square \end{aligned}$$

4. Pencils

Given a cycle $s \in \mathbb{P}^{n+2}$, a geometric s-*pencil* is a 1-parametric family of geometric cycles given by

$$\langle x_1, x_2, s \rangle \cap \Omega, \tag{8}$$

where $x_1, x_2 \in \Omega$ and the vectors $\{X_1, X_2, S\}$ are linearly independent. The dual geometric object

$$\mathbf{x}_s^\perp := \langle X_1, X_2, S \rangle^\perp \cap \Omega$$

is an s-*copencil*.

The matrix $[X_1, X_2, S]$ with columns spanning an s-pencil thus has rank 3, and determines an *oriented algebraic* s-*pencil* which we denote by $\mathbf{x}_s$ and which contains all cycles belonging to (8), while orientation is induced by the ordering of the pair (x_1, x_2).

An s-pencil is always nonempty since it contains at least the two spanning cycles x_1 and x_2. An s-copencil can be empty, can contain a single cycle, or an infinite number of cycles. The cycle s determines the type of cycles which constitute an s-copencil. For example, an r-copencil contains points and a w-copencil contains planes.

In the case of s-pencils, Proposition 2 can be improved in the following way.

Proposition 4. *Let $x_1, x_2 \in \Omega$ span an s-pencil $\mathbf{x}_s$. The corresponding s-copencil is empty if and only if $\Delta(X_1, X_2, S) > 0$.*

If $\Delta(X_1, X_2, S) = 0$, then there exists a proper cycle $y\omega$ such that $(Y \mid X) = 0$ for all $\mathbf{x}_s$.

Demostración. The case $\Delta(X_1, X_2, S) < 0$ is considered in proposition 2. In the case $\Delta(X_1, X_2, S) = 0$, there exists a nonzero vector $Y = \alpha X_1 + \beta X_2 + \gamma S \in \langle X_1, X_2, S \rangle$ which is a solution of the system

$$(X_1 \mid Y) = 0, \quad (X_2 \mid Y) = 0, \quad (S \mid Y) = 0.$$

Every such solution obviously also satisfies the conditions

$$(Y \mid X) = (Y \mid \alpha X_1 + \beta X_2 + \gamma S) = 0$$

for all $X \in \langle X_1, X_2, S \rangle$. In particular, $(Y \mid Y) = 0$, so y is an element of the corresponding s-copencil.

Finally, let $\Delta(X_1, X_2, S) > 0$. Since $x_1, x_2 \in \Omega$, the Lie form, restricted to $\langle X_1, X_2, S \rangle$ has signature (p, q), where q is at least 1. Since $\Delta(X_1, X_2, S) > 0$, it must be equal to 2. The Lie form restricted to the complement $\langle X_1, X_2, S \rangle^\perp$ is thus positive definite, so $(Y \mid Y) > 0$ for all $y \in \langle x_1, x_2, s \rangle^\perp$ and the s-copencil is empty. $\square$

A pencil is called *hyperbolic* if its determinant $\Delta(X_1, X_2, S)$ is negative, *parabolic* if it is equal to 0 and *elliptic* if it is positive. In view of the above proposition, we will mostly be interested in hyperbolic and parabolic pencils, where the corresponding copencil is nonempty, and determines new geometric objects.

Let us take a closer look at our two canonical examples $S = R$ and $S = W$.

An oriented r-pencil $\mathbf{x}_r$ is a 1-parametric family of cycles given by

$$\langle x_1, x_2, r \rangle \cap \Omega$$

and oriented by the ordered pair (x_1, x_2). If $\mathbf{x}_r$ is either hyperbolic or parabolic then, by Proposition 4, the r-copencil $\mathbf{x}_r^\perp$ is nonempty. It consists of point cycles which represent points in $\mathbb{R}^n$ in the intersection of all geometric cycles of the pencil.

If $\mathbf{x}_r$ is a parabolic r-pencil then, by Proposition 4, there exists a cycle $y \in \mathbf{x}_r \cap \mathbf{x}_r^\perp$ so the geometric r-pencil contains a point which is the common intersection of all cycles of the pencil. A parabolic r-pencil thus determines a point in $\mathbb{R}^n$.

If $\mathbf{x}_r$ is hyperbolic then $\mathbf{x}_r \cap \mathbf{x}_r^\perp = \emptyset$, so the pencil contains no point cycles. The corresponding r-copencil consists of point cycles lying in the intersection of all cycles of $\mathbf{x}_r$. The union of all such points is a *geometric subcycle*, that is a sphere or plane of codimension 2. The latter happens when all cycles in the pencil are planes, i.e. when $\mathbf{x}_r \subset \langle w \rangle^{\perp}$. On the other hand, if at least one cycle from the pencil is a sphere, the subcycle is a codimension 2 sphere. The subcycle is oriented in the usual way by the orientations of the geometric cycles given by x_1 and x_2 and by the order in which they appear.

An oriented w-pencil $\mathbf{x}_r$ is a 1-parametric family of cycles given by

$$\langle x_1, x_2, w \rangle \cap \Omega$$

and oriented by the ordered pair (x_1, x_2). The corresponding w-copencil $\mathbf{x}_w^{\perp}$ consists of cycles representing planes in $\mathbb{R}^n$ which are tangent to all geometric cycles of the pencil.

A parabolic w-pencils is spanned by two cycles x_1, x_2 such that $\langle X_1, X_2 \rangle$ contains W. The geometric pencil contains a plane which is the one common tangent plane of all cycles of the pencil. A parabolic w-pencil thus determines a plane in $\mathbb{R}^n$.

If $\mathbf{x}_r$ is hyperbolic then $\mathbf{x}_w \cap \mathbf{x}_w^{\perp} = \emptyset$, so the pencil contains no planes. The corresponding w-copencil consists of all common tangent planes of cycles in the pencil, and the points of tangency form a *cone* in $\mathbb{R}^n$. There is one exceptional cases. When all cycles in the pencil are points, i.e. when $\mathbf{x}_w \subset \langle r \rangle^{\perp}$, the pencil $\mathbf{x}_w$ represents a line in $\mathbb{R}^n$, and the corresponding cone degenerate to this line.

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