An everpresent reference of the XVIIth century intellectual production: Euclid's Elements

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Introduction

Euclid's Elements, written in the third century BC., during the reign of Ptolemy I, is, second to the Bible, only in number of reprints. Those published in the XVIth and XVIIth centuries are, either philological examinations of the corpus, or the basis of philosophical commentaries, or mathematical treatises or even pedagogical handbooks. Among those very numerous reprints, I shall only quote from the version offered by Christopher Clavius, which, as far as we know, was referred to by all XVIIth century specialists.

This Euclidean reference fed equally all fields of intellectual production, whether philosophical, mathematical or pedagogical and the Elements was the common, cultural inspiration of mathematicians and natural philosophers from the end of the XVIth century to that of the XVIIth.

1. The Elements and the Switch to Mathematical Physic

Now, what part do the Elements play in the basic inspiration of the scientific revolution, that is the use of mathematics in natural philosophy? All the authors we shall mention do cherish the "new" concept that the laws of nature are to be discovered and enunciated in mathematical terms. Thus, we have to examine Galileo's case. The adequation of geometry to material phenomena is undoubtedly for him the criterion which distinguishes the various philosophical attitudes; such is the stake (l'enjeu) disputed between Simplicio, the champion of Aristoteles in his *Dialogo*, and Salviati and Sagredo, the modern scientist and the man of culture. Clear is the program, but most difficult its actualisation. Galileo's physics is, obviously, determined by what mathematics are at is disposal, and, conversely, mathematical notions re elaborated trough the basic concepts of the new natural philosophy. Now, what mathematics can he use, if not, above all, Euclid's *Elements*? But it so happens that Euclid's work has not been conceived as a language permitting an immediate dialogue with natural phenomena, but as the foundation of a pure, abstract science. Thus, the Elements deals with such quantities as numbers, lines, surfaces, volumes and angles, while Galilean (and post Galilean physics) concerns ponderous bodies, time, speeds and densities.

Such is the context to grasp Roberval's warning to the reader when opening Book IV of his *Elements of geometry*: "there may exist a ratio between two objects which you do not compare as quantities: such is the case if you compare weights, lights, sounds, motions, forces and other attributes. But this ratio does not belong to pure geometry, which only includes the five modes we have mentioned [. . .] All other ratios belongs to physics. Still you cannot but use them in the mixed mathematics of physics, as in optics, in mechanics, in music, astronomy, etc. which are not exactly relevant here, since we intend to restrict our examination to geometrical ratios. But it must be kept in mind that the very same method we shall use in these present work will be of the same utility when we have to study other ratios. Thus, this warning is but meant to exclude those ratios which do not belong to geometry and which, accordingly, will be dealt with in some other place."

The natural philosophers in the steps of Galileo, mean to exploit, to carry over the solutions of pure geometry to fields which those notions did not initially concern. According to the Aristotelian tradition, mathematics is not a science of material objects, but nor is it a science of objects actually existing apart from material objects (it is not a science of ideals objects). What is posits (suppose) as distinct has no autonomous existence in nature. Besides (en outre) -and frequently enough- that nature you want to express in mathematical terms, offers but changing phenomena and functional relationship between those "quantities".

Galileo can hardly find, in traditional mathematics, tools to account for those relationships. Euclidean mathematical tools, which Galileo cannot but refer to, offer "too meagre a design to satisfy the demands of physics"and can serve only a very elementary physics.

Galileo had intended to add a fifth and sixth "day"to his *Discorsi e dimostratione matematische sobre due nuove scienze*, the sixth one -never finished- was to study percussion and the fifth the theory of proportions. It was not included in the treatise as published in 1638, but composed a short time after with the help of Torricelli.

I shall mention but one of those elementary but taxing obstacles he faces: two quantities can bee proportional only if they are homogeneous (line and line, number and number etc.). This is an essential feature of the theory, which we just forget when we identify the measure of a quantity and the quantity itself - a measure being nothing but a proportion. Thus, it is not possible to symbolise directly a physical quantity -speed for instance, as the ratio of a space and a time interval which are heterogeneous; another example would be a specific weight as the ratio of a weight and a volume.

The constraints imposed on physics by those presently inefficient tools bring some "light"on a precise and vexing point concerning the theory of falling bodies. Two developments, one from Galileo, the other from Descartes, challenge the reader as both contain a "mathematical error"in their use of Euclidean proportions. Let us conclude that Euclidean mathematics and physical concepts do not mix peacefully. One will thus keep in mind that the developing program to account for natural phenomena through mathematics will, on one hand enlist and stimulate Euclidean geometry, and on the other hand demand new concepts.

2. Demonstrative Order and Prime Propositions

Among Euclidean propositions, some are not deduced from antecedents: they are either common notions or axioms, demands (the so called postulates), or definitions. The XVIIth century writers criticize the Elements from two opposite point of view. For some, Euclid has defined too many terms and demonstrated too many propositions; for the others, the flaw is the lack of definitions and demonstrations.

We shall then give some examples of a first category of criticisms:

1. The third common notion ("when substracting equal things from equal things, remainders are equal") can, and thus must be demonstrated.
2. The definition of a straight line ("a line is straight when it runs evenly through the points it contains"-Book I, def. 4) is not sufficient, for it wants a statement of existence- not to mention that the very choice of the definition is ambiguous.
3. The construction of an equilateral triangle ("on a limited straight line, construct an equilateral triangle", Book I, prop.1) wants proofs, as the demonstration posits that the two auxiliary circles intersect, and thus postulates some principle of continuity.
4. The property concerning equal circle arcs (Book III, prop. 28) should be extended to the examination of unequal arcs. On this point -and many others- Euclid does not satisfy the demand for complete and general solutions.
5. Even though (sou) (bien que) the presupposition of a total order on finite (fainait) homogeneous quantities may be detected in Book V, prop. 9, 10 and 13, no statement (were it a postulate) exists to back it.
6. At last, it would be possible (and thus necessary) to start from a better definition of the relation of order, so as to demonstrate why the whole is larger than the part (a property prosecuted as the eighth common notion).

On the other hand, a second category of criticisms points at the uselessness of such statements: the straight line has no need of a statement of existence, but what is more, its very definition is unsuccessful, or even harmful (nuisible), as it creates confusion about a notion of perfect clarity *per se*. To Roberval, who, in a lengthy sequence of statements, propose a definition of a straight line as the invariant ensemble in a rotating solid, Arnaud opposes the following approach: “The notions of a flat surface and of a straight line are so simple that you would only obscure them trying to give definitions. The one thing which is required is to give examples so as to have the notion inlaid into the words of different languages”. Total is the disagreement and the definition of a straight line through the invariants initiates a geometry founded on the properties of isomorphous transformations, a geometry that refuses intuitive or natural notions.

The construction of an equilateral triangle is useless, as it is a self evident truth, well-known to all. In the same way, the property concerning equal arcs and their sub tenses is intuitively grasped and does not require a demonstration; as to the order of quantities, it raises no problem and is self-evident. Finally, that the whole is larger than his part is a prime principle, so obvious that it does not need a demonstration (on that point, Arnauld agrees with Euclid).

To persuade (pirsweid) ourselves that the two approaches were simultaneously at work, we have only to read the treatises of Roberval advocating the first line, and those of Arnaud, championing the second. Both men, at the same time, ascribe themselves a similar task: to examine Euclid’s Elements and reform them. They express similar diagnoses: the Alexandrine’s work is imperfect, an even unfit to impart satisfactory bases to those who intend to venture in the field of geometry. Antoine Arnaud has the first edition of his *New elements of geometry* published in 1667, while Roberval starts writing the last version of his *Elements of geometry* in 1669. According to the first author, Euclid’ *Elements* ignores the natural order and clear ideas; for the second, the flaw is that the demand for demonstrations is not pressing enough. A comparative study of their *Elements of geometry* reveals how much Arnaud means “to give fewer demonstrations”(than Euclid) whereas Roberval intends “to give many more”.

2.1. Intuitionism

“It seems to me absolutely fruitless to look far and wide for proofs to something we cannot question, provided (pourvu que) we pay some attention to it”, Arnaud writes. To demand too many demonstrations is a mistake and would ignore the necessary humility of the human understanding: "geometers, being more prone to convince than to enlighten the mind, believe they will be more convincing by inventing some proofs, even of the most obvious points, than by merely stating them and leaving to the mind the care to acknowledge its evidence".

Such a doctrine (doctrin) is developed at length in the *Logic of Port Royal.* Descartes partly inspired this very critical approach, and we recognise an effect of the theory of clear and distinct ideas. As against a steril logics, he stresses knowledge is acquired through intuition and deduction. Definitions, axioms, postulates, are accepted as warranted by the "mind's vision". The axioms of elementary geometry are eternal truths; indeed, God might have created different ones, but we cannot but submit to what they teach us. "Descartes retains as a mood the unquestioned evidence of Euclid Elements", wrote Yvon Belaval.

Pascal's position is less abrupt:

"Geometrical knowledge does not define everything, nor does it prove everything, but it works only on things which natural light renders clear and permanently true. This knowledge does not demand that everything be defined and demonstrated, but to keep half-way, that is not to define those things that are clear and understood by all men, but to define all other things. Do sin (pécher) equally against the essence of geometry those who undertake to define and prove everything, and those who do not care to do it when things are not self-evident [...]".

The Pascalian way of dealing with Euclid amounts to a sort of *chiaoscuro*. He admires and respects this masterpiece, he sees it as a model of the asymptotic proximity between geometry and the perfect method of knowledge. Meanwhile, he agrees to the necessity of a reorganisation and reform of the thirteen books. One cannot account for (expliquer) Pascal's approach except one keeps in mind his first aim is not to found the absolute certainty of knowledge, but in the process to improve his own power to convince. Mathematical argument and reasoning may not be seen apart from similar processes used in treatises dealing with ethics, apologetics or physics. The art of persuasion is identical in all these fields. Indeed, it appears in a purer -simpler- form in mathematics, but it would be a mistake to think it is more specific in that field.

The conceptions of Descartes', Arnaud's (and to a lesser degree of Pascal's) converge to spare them (leur épargner) the search for a "general economy"of first principles as related to one another. Each author, in his own way, concerns himself with the unquestionability of the principles, and leaves aside their organisation as a system of logics.

3. The Urge to Demonstrate

Roberval takes a firm stand against the previous approach. "Each thing can be demonstrated, whatever clarity or evidence it may seem to have by itself, as there exist other obvious truths which may be selected and utilized to demonstrate the first one. The reason for this is that, since it can be demonstrated, it cannot be styled a principle, since principles may not derive from other principles, or any other statement, but must be, each for itself, grounded on itself only. Thus does a science achieve perfection, in that it is simpler, since it is grounded on as few evident and undemonstrated principles as possible. More generally, any mathematical truth which is not a principle must be demonstrated; otherwise it cannot be received, being but an indefinite part separated from the whole it belongs to, which would break the unity of science, a unity which must be preserved at all costs"

Roberval thus considers the perfection of science as endangered by the lack of demonstrations, while Pascal and Arnaud saw that danger as coming from an excess of demonstrations. For Roberval, the intrinsic true of statement is no criterion of their status: his geometry is not a collection of true statements. What is essential is the optimisation of the structure as one whole, the return of the "deductive input".

The "intuitive", or evident, simple, immediately clear, aspect of prime principles loses strength, as their value as hypotheses is strengthened.

What reform had been advocated for axioms and postulates is extended to deal with definitions. Roberval bears three reproaches against Euclid.

1. Many of his definitions are not explicit: such is the case with part, length, limit, breadth, gradient (inclinaison), equality, bigger or lesser, continuity, interval, whole, area, quantity, measure (mizer), oneness, plurality and depth.

2. Other ones are ambiguous and want strength, as evidence by the three definitions of the ratio (def.3, 4, 5 in Book V).

3. Finally, several definitions are never backed by a statement of existence that would confer them what actually would legitimate a demonstrative process: definition of line, surface and plane (2,5,7, Book I) of the part and of the multiple of a quantity (1 and 2, Book V), of the unit and the integer (1 and 2, Book VII), of the solid (XI,1).

4. Geometry Reduced to Logic

Later in the century, a same sort of approach has to be mentioned, but in an extreme form: Leibniz, repeatedly stressed the necessity to define and demonstrate beyond an unquestionable evidence. Sound (saine) is the method to stake principles whose certainty is not absolute and then to deduce necessary consequences: whether these be true or not, the deductive link in itself adds knowledge and thus, legitimates the process. From this point of view, Euclid style is praise worthy, and it is fortunate that the ancients and especially Euclid "decided to push forward before demonstrating these axioms they were obliged to use". In this first respect, the Euclidean corpus is to be admired for the organisation of its logical and deductive structure. Those logical demands made it possible to go beyond empirical geometry, which is advantageous in terms of method, but from a practical point of view as well, since one may demonstrate properties which cannot be discovered by intuition nor imagination, such as asymptots or the finite measure (fainait mizer) of some areas of infinite extension.

Such results, though, does not imply one should not proceed much further -upstream, so to say- in the analysis of principles than ancients as Euclid, and possibly than contemporary mathematicians have done. Examples borrowed from Euclid are frequently used by Leibniz to show it is possible to reduce the axiomatic bases through the demonstration of explicit or implicitly accepted statements: part and whole, subtraction of equal quantities from equal quantities, the fifth postulate, intersections of circles, the existence of the plane: similarly, it is not the "clarity"of the expressed concept which legitimates a definition, but the demonstration that the defined notion is possible, i.e. non contradictory.

"I am convinced"Leibniz writes, "that to make sciences more perfect, it is even necessary to supply demonstrations for some propositions known as axioms, in the way Apollonius get to demonstrate some of those which Euclid had left undemonstrated. Euclid was right, but by how much more Apollonius. It is not indispensable to do it, but of some importance and even necessary in some regard. The late M. Roberval had in mind his *Nouveaux éléments de géométrie*, in which he meant to give close demonstrations to several propositions Euclid stated or implied. I do not know whether he saw the end of his work before he died, but I do know many people laughed at him; if they had know the importance of the matter, they would have had second thoughts. It may not be necessary for apprentices, nor for the common pedagogues, but to advance sciences and sail past Hercules' columns, nothing is more necessary".

Leibniz logical project amounts to reducing all those fields of knowledge ruled by an absolute or metaphysical necessity (geometry being one of them) to one and the same principle, that of identity. All demonstrations, as well as definitions could be deduced from that one statement, *A is A*. Basically, all other axiomatic forms have but a temporary justification.

Speaking of Leibniz, one has to determine the part of Euclid's Elements in his ambitious project of *Characteristica Universalis* which was to be an absolutely explicit and univocal language, allowing some sort of an automatic computation of concepts. "If mathematics, geometry, algebra, arithmetic (even latest developments and infinitesimal methods) constitute but one field of the great formal science of order, they agree so well with the *Characteristica* that they will supply quite a share of the paradigm of the *Characteristica Universalis*. It is to geometry that *Characteristica Universalis* applies best -at least most immediately".

No element of Euclid's geometry would be posited in the *Characteristica geometrica* "for it allows to rediscover, through that kind of computation everything that geometry teaches, even to its elements, in an analytical and determined process". That over-ambitious project survives only through some simples, somewhat elementary, examples and not immune to criticisms.

The reading of Euclid by XVII$^{\text{th}}$ century authors thus draws a real bifurcation between mathematicians leaning (s'appuyant) on intuition and self-evidence (Arnaud, Descartes) and mathematicians prone to ground their science in axiomatics (Leibniz, after Roberval and Pascal). "What Descartes initiated was that mathematical option which deems (*juge*) ridiculous, or nearly so, the critical approach to axioms; what Leibniz prepared was the road to axiomatics ", writes Belaval. One thus observes the rediscovery of two strong traditions of geometry, proceeding from those interpretations of Euclid's: one aims at constituting this field as a positive, or even natural science, the other means to found it as hypothetical and deductive.

5. The Straight Line and the Theory of Parallels

As could be expected, the theory of parallels and what pertains to the fifth demand are good indicators of the approaches to Euclid in the XVII$^{\text{th}}$ century. J.C. Pont, an historian of mathematics, has written that "since the epoch-making commentary of Proclus, one answer has crossed centuries, handbooks, works dealing with the history of geometry: the fifth postulate is not evident enough to be vested with the dignity of an undemonstrable proposition". Thus, the choice has always been either to demonstrate it, or to demonstrate an equivalent proposition (which amounts to the same), or to posit another and more acceptable postulate so as to reorganise the axiomatic bases and transform the former postulate into a theorem"

No wonder Arnaud is found among the many ones who denounce the "mistake"to demonstrate so obvious a proposition. Arnaud organises his chapter on parallelism according to a sequence of "natural"demands; six axioms are enumerated at once, the sixth one reading "two straight lines, when extended on the same side, will get closer to each other, and finally intersect [. . .] Euclid sees this proposition as a principle, and rightly so; for it is clear enough to satisfy the understanding, and it would be a waste of time to rack one's brains to prove it through circuitous reasons".

As to other propositions and results concerning parallels and the fifth postulate, they are deduced from the property, given as an axiom, that the normal to the middle of a segment is the set of points equally distant from the two extremities. To justify this axiom, Arnaud only writes "I contend that to consider the very nature of the straight line proves the truth of this proposition".

On the other hand, XVIIth century authors will join those who mean to demonstrate it. It had been supported that it was the intention of Euclid himself, for it is possible to retrace his many endeavours (indevors) to downgrade, as much as possible, the postulate in the *Elements*.

Euclid defines parallels as "straight lines that will not meet"(I,23). Critics first used an other definition of parallelism, and specially the following one: "parallel straight lines are equidistant ones". Such a definition was favoured by some of the major Ancient and then Arabs geometers (Posidonius, Aganis, Tabit-ibn Qurra or Ibn al Haytam). This definition, if supplemented by the exhibition of two such lines is known to be equivalent to the fifth postulate, whose demonstration is then possible, starting from the definition through equidistance and a statement of existence.

Some authors, intending to demonstrate the celebrate postulate, will thus concentrate on the existence of equidistant straight lines. Roberval achieves this aim, trough his famous quadrangle including three right angles. It has been shown, of course, that the whole process rests on an axiomatic different from Euclid's, but finally equivalent. This intense scrutiny of the postulate status and the close examination of the underlying axiomatics will be most instrumental in the initial steps towards non-euclidean geometries (especially in the works of Saccheri and Lambert).

As Henri Saville had remarked, as early as 1620, "there are two blemishes in the body of geometry. One of them being the fifth postulate, the second one being the ungeometrical and useless definition of compounded ratios". The second cause for those sleepless nights is what I will presently address.

6. The Theory of Proportions

As fireworks displaying their utmost magnificence in a final bouquet, this theory displays its utmost potentialities before it disappears, displaced by the automatic process of the algebraic calculus. The theory of proportions, inherited from Euclid's *Elements* appears to XVII$^{\text{th}}$ century mathematicians as indispensable, but very core of their procedures of demonstration. Meanwhile, the attempts to complete, modify and enlarge it, multiply. Three types of obstacles inspire and account for such attempts:

1. Those which are related to long-known internal deficiencies: the comparative and not self-operating nature of the ratio, the want of a total order of ratios, the implicit acceptance of the fourth term in a proportion, as well as the twofold handling (traitement) of quantities, depending on whether they are continuous or integers.

2. Those which are related to a broader concept of numbers, an age-old notion but which will be more precisely actualised in the XVIth and XVIIth centuries; such an evolution is more specifically connected with the general switch to algebraic methods.

3. Finally those which derive from the demand to incorporate infinite (infinit) sets of ratios of infinitesimal quantities. The use of mathematics by the new natural philosophy required such modes of calculus; more especially, it is the central point in the Galilean, and then Newtonian cinematics.

6.1. Relation or Quantity

The Euclidean ratio, properly speaking, is neither a number nor a quantity; still in the original text, it takes in many ways after a quantity: it is shown to be equal, or unequal, or to lend itself to operation-like transformations.

The research to give ratios a mathematical status allowing operations (and thus not conceiving them as relations only) is given a new impetus when the Euclidean text is re-examined; thus does Clavius breathes new life in the theory of *denominatio*. To any ratio is associated a *denominatio* while is but "that number which exposes clearly and openly the way one quantity relates to another". This somewhat vague concept was expected to allow ratios to be compared, to be composed through the multiplication of *denominatios* etc.

Clavius is certainly wrong when he claims his interpretation is the only one true to Euclid, and that all authors who differ betray the Alexandrin geometer. Many arguments can be mustered against the existence -be it implicit- of such a notion as the multiplication of ratios in Euclid's work. There are but two results (VI.23 and VIII.5) in the whole *Elements* connected with composition of ratios. The refusal to identify the composition of ratios with a multiplication of quantities is a firm stand, and championed till a late period. Many authors, Newton one of them, consider that ratios cannot be composed as numbers are, and that they do not partake of the same nature as quantities.

6.2. An Attempt Towards a Reform: Arnaud

The doctrine of proportions raises a touchy point for those who consider intuition and clear ideas, or evidence, to be the foundations of mathematics. Indeed, one cannot state intuitive truths -clear and distinct ones, that is -concerning ratios and the equality of ratios. On that question, Arnaud is "trapped". While the theory of proportions is very much present in his geometry, it works uneasily and painstakingly. He ignores the definition of the Fifth Book, possibly because he deems it too distant from the criteria of evidence - in what he is right.

Arnaud, who had intended to simplify the prime concept of the theory is unable to avoid deadlocks (de se sortir de l'impasse) , as shown by his warning XXIX: "What notion I propose of equal ratios would be sufficient, were it always possible to appreciate whether antecedents are to be found equally in the derived quantities -which, most often is uneasy"

He then initiates a second definition of equal ratios, which definition, "one of the most difficult in geometry", also has unexpected consequences and heads to contradictions. J.L. Gardies, when examining Arnaud's attempts, discovers unsteadiness and awkward (maladroits) uses of suspiciously selected instances. Intuitive clarity gives birth to confusion and Arnaud, in the following printings of *Nouveaux éléments de géométrie* will use a definition much closer to that of Book V.

6.3. An Orthodox Endeavour to Reexamine Matters: Roberval.

As many contemporaries, Roberval forsakes the division of the theory of proportions into two parts (quantities and numbers). Numbers are quantities as well, and the theory of Book V must be made valid for numbers too. Apart from this point, his proceedings are the most orthodox in this period. He discards such solutions as might ignore the difficult points of Euclid's theory, the one which proposed a definition of the ratio different from Euclid's, and the one which extended numbers so as to include continuous quantities and even ratios. He remains true to Euclid -allowing for important "nuances"in his definitions of numbers and ratios. His intention is to fuse the arithmetical books of Euclid into Book five.

Numbers, multiples and divisor of numbers, Euclid's Algorithm, all the specifics tools of arithmetics are included in his book VI about proportions in general: an original and successful attempt.

Roberval stressed the necessity to arrange in order the whole set of ratios, no easy task. Thanks to his definition and by using the order existing among equimultiples, he demonstrates that such relations as "bigger than"or "lesser than"are reciprocal, and for ratios as well. Appreciating Roberval' s results, one may deem that the logical structure of ratios is far better argumented in his *Elements* than in Euclid's and allows for a more efficient operational use of the theory. Still, as he declines to assimilate ratios to numbers or quantities permitting internal operations, the difficulty increases, so that the logical structure, even though strengthened, cannot escape some sort of ghost-like quality: ratios still share an ambiguous status between relations and quantities.

6.4. An Attempt to Reach a Conclusion: Grégoire de Saint Vincent

What is lacking in Euclid's theory which would enable it to solve geometrical problems, whether old or new? Very little, is the answer of the Jesuit Grégoire de Saint Vincent. Committing a blunder (gaffe) he will atone for a long time in the purgatory of underrated scientists, he publishes a big treaty which so perfectly completes the theory of proportions that he claims it can bring a solution to the oldest and most stubborn problem, that of squaring the circle. The *Opus Geometricum* of Gregoire nevertheless remains an important work. First because you can find therein brilliant ideas to be developed in later years, such as the squaring of the hyperbola; but also because he exploits the almost possibilities of Euclid's Book V. The author is aware his project will face scepticism: how could such an antiquated (vieillie) theory supply now what solutions it has proved unable to propose for so many centuries? So, he claims a little "push"is necessary:
"Such is the reason why I thought it good to discover new ways and new methods which could fill the wants of ancient geometry -namely by looking for the magnitude, that is to say the quantity, of any ratio, if you see it as indefinitely [*ad infinitum*] extended".

As observed by Jean Dhombres, "it really is a conclusion to the theory of proportions that Gregoire supplies by actualising unlimited geometrical progressions. The "term", that is to say the sum of the geometrical progression under study, is basically, the very term of Euclid theory in book V and the operational target of the *Opus geometricum*. Let those remarks suffice (se-fais) and let us remember that this *ad infinitum* push, once given, is made compatible as things go, with Book V, so as to become an internal re-ordering of the theory. Gregoire attempts to conceive the limits of series as partaking of the Euclidean doctrine.

6.5. Extending the Notion of Number

Others authors lead XVIIth century mathematicians toward an extended notion of numbers. Thus the *Encyclopédie Méthodique Mathématique* of Diderot-d'Alembert stressed that "M. Newton defines more precisely a number, not as a collection of units as does Euclid, but as an abstract ratio of a quantity to another of the same nature, taken as a unit ; following such a notion, he divided numbers into three categories : namely integers, containing the unit so many times and without remainders, for instance 2,3 4 ; broken numbers or fractions and deaf numbers, that is incommensurable ones". In fact, Simon Stevin in his master book *La Disme*, had paved this track (prepare le terrain) which Malebranche, Wolff, Leibniz will follow.

6.6. The Potentialities of Algebra and the Theory of Proportions: Descartes

The growth of algebra, inherited from Arabs by Italian scientists on the XVII$^{\text{th}}$ century, and then transmitted to the other European countries and into the hands of such people as François Viète, Thomas Harriott, Nicolas Chuquet, dismisses the theory of proportions to its grave. Obviously, once the transcriptions, the symbolism, the rules to multiply and transform polynomials and equations have proved their efficiency, the central concepts of Eudox and Euclid 's theory are but dead instruments. When the habit is taken, for instance, to equate $a/b = (c/d)^3$ and $ad^3 = bc^3$, or to recognize in $y^2 = 2ax - x^2$ the equation of a circle, then the quaint (bizarres) manipulations of ratios and proportions will be replaced by ready-made techniques of computation; very soon the (Euclidean) whys and hows of such transformations will be forgotten: in our first example one will not keep in mind the notions derived from the definition (V,10) of doubled or trebled ratios, nor in the second example suspect the presence of the proportion $y : x :: (2a - x) : y$, where geometrical interpretation identifies a circle.

It should be noted, though, that the founder of algebraic geometry do rest their new technique on the theory of proportions. The one example of R. Descartes will prove the point.

The first advance by Descartes is to transform the product of two quantities into an internal operation. Traditionally the product of two lines was the rectangle built from them, but while the result is perfectly determined, it is not an internal operation since this result (an area) is not homogeneous with the initial data (lengths). Descartes does not evade the question and explicitly warns he does not intend to multiply two lines. If one cheeks carefully, the *Geometry* offers no instance of an operation which would associate to two lines their product-line. To three lines, one of them styled “unit”, it is possible to associate a fourth one, defined by the theory of proportions and Thales proposition (similar are the cases of the division and that of the square root, the later being based on a corollary of Pythagora s theorem).

Indeed, a and b being given, $z = a \cdot b$ is but another form of $z : a :: b : 1$. Thus, z is homogeneous with a, and not with a.b. It follows that to write $z = a \cdot b$ is meaningless unless you keep in mind it signifies $z \cdot 1 = a \cdot b$. Still, the new technique is profitable, since you can carry on the operation, whatever the number of dimensions. The *Geometrie* is not only the land-mark of the nascent algebraic geometry, but also, and possibly more essentially, that breaking point in the history of mathematic when the theory of proportions yields precedence to algebraic calculus through a broadened notion of quantity: the ratio inherited from Eudox and Euclid is transformed into a quantity with which and on which operations become feasible. It is well-known that in the theory of proportions, genuine operations, if any, are restricted to the set of rules allowed for preservation of proportions. Descartes' *Geometrie* is undoubtedly derived from Eudox and Euclid's grand doctrine of continuous quantities, but it successfully grafts on it a neutral element and a multiplication. Descartes mathematics are rooted in Euclid's doctrine, exclusively, but will fructify well beyond what could be conceived in the thirteen books of the Alexandrine scientist. While getting rid of the “thorniest”points of the doctrine, he retains the core of it, and one cannot overstress the notion that the Cartesian algebraic geometry is centred on the theory of proportions: a point to be more carefully examined at a later stage.

It is unquestionable that the Euclidean origin of Cartesian mathematics sets a limitation since, in so many words, any relation which cannot be expressed in terms of perfect ratios is unfitted for mathematical knowledge: such is the case with transcendental quantities. Still, we must ask a question: why does Descartes's use of proportions outdates the theory, which is not the case with Gregoire, and Roberval? We know the answer already: while Gregoire, and Roberval as well, intend to built their algebras on ratios, Descartes assigns as essential part to the algebraic symbolism of roots and loci, and thus launches the new geometry as an algebra of polynomials -what algebra we now use. The one specific re***ant of the superb Euclidean machinery in the Cartesian system is the rule of three. Let's just keep in mind that the algebraic automatism that will discard the Euclidean method to transform equalities and inequalities derives initially its justification from this very method.

6.7. Two Attempts at Generalisation: Cavalieri, Mengoli

At first sight the theory of proportions seems unfitted for many new branches of XVII$^{\text{th}}$ century mathematics. To be connected by a ratio, the quantities must be of the same nature (3$^{\text{rd}}$ definition) and must, when multiplied, be compared as smaller and larger (4$^{\text{th}}$ definition). When Cavalieri wrote his *Geometria indivisibilibus recondite*, he introduces ratios between quantities expressed through indivisible terms; now, it is obvious that such terms as used by Cavalieri do not satisfy the demands we have to far mentioned: they are not Euclidean quantities. Galileo's disciple will nevertheless try to invest his theory within the conceptual framework of Euclid's Book Five.

He attempts to define a new type of quantity, neither a line, nor a surface, nor a solid, but a *collection of indivisibles*: a figure being given on a plane, the geometer defines a *Regula*. A plane parallel to this *regula* is motioned from one tangent to the opposite tangent. The intersections between plane and figure (which are the indivisibles of the figure), when considered together (*communes sectiones simul collectae*) are known as *Omnes lineae*. The central object now is the study of those ratios which may exist between the different collections of the *Omnes lineae* of two figures.

One essential principle to work out the properties of the *Omnes* is as follows: "*Ut unum ad unum, sic omnia ad omnia*". Basically, it may be interpreted as the transition from one indivisible to another and one figure to an other through the ratio of one *Omnes* to another.

Granting two figures F_1 and F_2 intersected through a common rule, Om_1 to Om_2 naming the collection of indivisibles, the following conclusion is to be reached:

$F_1 : F_2 :: Om_1 :: Om_2$, which is Book II, theorem 3.

Cavalieri is perfectly aware he creates a new mathematical object, and carefully refrains from identifying F_1 and Om_1 as one and the same object. He must promote this new object to the status of a quantity, as defined by Euclid, meaning it must conform to the axiom of Eudox and Euclid and Euclid's definition (V,4). *Omnes lineae* must be compatible with equality, comparison and order axioms. Such is the meaning of Book II, theorem 1: "*Quarum libet figuram omnes lineae sunt magnitudines inter se rationem habentes*" *(from which does appear that "all the lines"of the plane figures are quantities to be compared through ratios)*. Here is the principle of indivisibles as summed up by Ettore Caruccio: "*If two plane figures are intersected by a collection of parallel straight, and if the chords defined by the intersections are equal, then the areas of the figures are also equal. If the chords have between them a constant ratio, so have the areas. Similarly, if two solids are sliced by a collection of parallel planes and the areas between such planes are equal, so are the volumes. If the areas have between them a constant ratio, so have the volumes*".

Cavalieri's theory is thus to be considered as an attempt to conciliate infinite collections of objects and Euclid's rules -that is an attempts to extend those rules, so as to satisfy the demands (to quadrate curves, to locate gravity centers, to clarify the notion of speed) of contemporary physics and mathematics.

Mengoli, Cavalieri's disciple and heir to the chair in Bologna, develops a broad theory based -so he says- on Euclid's fifth book, which he calls the theory of near-proportions. His intention (not unlike to Gregoire's) is to stretch the validity of proportions to the limit. He thus defines the near-infinite ratio as that which can be made larger than any given ratio, the near-zero ratio as that which can be made smaller than any given ratio, and the near-equality of ratios when the difference between two ratios is smaller than any given difference. His intention then is to validate the theory of proportions when operating on limits.

The debt owed by infinitesimal calculus to the methods based on proportions is acknowledged one century later in the *Encyclopedie Mathematique Méthodique* (Vol. I, p.703), which reads: "differential calculus is essentially the ancients' method of exhaustion, compressed into a simple and practical analysis: it is the method to define through analysis the limits of the ratios".

6.8. Newton and the Principia

How entrenched (a résisté) Euclid's doctrine stood appears in another way in Newton's work, as he utilises ancient methods at the very moment he creates new ones.

The mathematics practiced in the *Principia* attempt to do without infinitesimals and even the algebraic approach, and to reintroduce classical geometrical methods. It is centred on the *first and last ratios* and evidences Newton's confidence in Euclid's theory.

Commentators could not but puzzled, and it is known that from the beginning of the XVIII$^{\text{th}}$ century, the editions of the *Principia* were supplemented by algebraic and infinitesimal versions of Newton's demonstrations (Clairaut's is a case in point). Recent studies of Newton's intellectual evolution open new vistas to our presentation. Their common point is that Newton's mathematical acquisitions have not followed the historical order of the growth of mathematics. He started as an accomplished adept of Descartes' approach to algebra and to its relationship to geometrical constructions. As far as we know, he did not, in his younger years, attempt to enrich the ancients' *treasures of analysis*. On the contrary, from 1673 to 1683, while he was Lucas Professor in Cambridge, he discovered them. He may have, at the same time, gathered doubts of, and objections to, the potentialities of Cartesian geometry. Then he re-examines the very roots of classical and geometrical methods. Thus will he choose to write the *Principia* not in the semi-classical style of Descartes, but in truly classical and Euclidean style.

That future style of the *Principia* has been worked out in a critical perspective, as evidenced in the 1680 treatise of *Geometria Curvilinea*. He at once exploits classical treasures and works out completely new techniques, especially a mode of calculus centred on the concept of function. Massimo Galuzzi sums up Newton's stand as follows: "He attempts to graft the advances of calculus onto the trunk of classical geometry [...] To put it briefly, *Geometria curvilinear* could be described as an attempt to generalise the classical theory of proportions".

One must stressed at that point that the royal avenue is still that which have been open by Euclid's Book V, in the very period when a conceptual remodelling leads mathematics toward infinitesimal calculus, and in the work of the very author who blazes (pose les jalons) the new trail.

My intention, in this last points, has been to show that all those mathematical techniques aiming at manipulating infinite quantities (indivisible parts and limits) are promoted by authors who claim a total allegiance to the theory of Book five, may they at times reform it.

All this is but a swansong, though. Algebraic methods, the advances of infinitesimal analysis for ever supersede the theories of Eudox and Euclid's. The rules of algebraic calculus, the technical control of limits, the introduction and development of the concept of function, transcendental equations and curves soon outclass the methods of Book V, which Leibniz and his heirs no longer use. Leibniz immediately forges (and has it adopted by the *Acta eruditorum*) unified symbols for ratios and divisions, since a proportion is written $a : b = c : d$. He is even more explicit in the following passage:

"I have always stood against the use of special symbol for ration and proportions, as the division sign is sufficient for ratios, and that of equality for proportions. As a consequence, I write the ratio of a to b this way, $a : b$ or a/b, as you would to divide a by b. I inscribe a proportion, which is the equality of two reasons, as the equality of two divisions or fractions. Thus when I mean that the ratio of a to b is the same as that of c to d, it is enough to write $a : b = c : d$ or $a/b = c/d$".

Conclusion

That Euclid was ever present in the mathematics and physics of the XVII$^{\text{th}}$ century, has been shown to satisfaction -so I hope, at least; but it is a puzzling presence, and announces some sort of fading off (affaiblie), if not of a crumbling down. Euclid's works are also present in another field at the same period, more essentially and less critically that in sciences. I mean the field of philosophy. This second presence will just be mentioned, but it is most impressive. Mathematics certainly is the model to accede to the truth that Descartes, Pascal, Hobbes, Spinoza, Leibniz, and so many others cling to. Of course, different authors define different processes: the concepts of certainty vary, the transfers of those mathematical certainties to natural phenomena, to moral arguments, to metaphysical demonstrations are seen as more or less feasible; but for all authors the model does work. For Descartes, the sparkles of divine truth are to be found in geometry. For Pascal, while the perfect method never to err is beyond human reach, the best and most certain method accessible to men is perfectly taught by geometry. In order to conceive clearly the modes of perception leading to certain assertions or negations, Spinoza uses "but one example" and precisely, that of four proportional quantities, to conclude that only mathematicians conversant with the theory may gather adequate notions. As for Leibniz, he writes metaphysics when he writes geometry and reciprocally. Now, for all of them, apart from legitimate technical criticisms which one may and must direct at the doctrine, Euclid's geometry remains the model of the model.