Extract of the anatomica by René Descartes February 5, 1635

Vincent Jullien
Profesor Universidad Bretagne Occidentale
Bretagne Occidentale, Francia
vincent-jullien@wanadoo.fr

If a body were to be pushed or received an impulse from a force applied evenly by a spirit (in effect, no other type of force could be as even), and if it were to be driven in a void, it would always take three times more time to travel from the beginning of its movement up to the midpoint of the space to be traveled than from the midpoint of the space to its end, and so on.

For because no void of this type can be created and because, whatever the existing space, it will always resist in some way: as such, the resistance always increases in geometric proportion to the speed of the movement, such that one ultimately arrives at the point where the speed is not perceptibly augmented and it becomes possible to determine a certain other final speed, to which it will never be equal.

Just as gravity is never evenly applied, as the soul would do, bodies submitted to the impulse of the force of this gravity, but as there is a certain other body already in motion, not only can it never put a falling object in motion as quickly as it is itself being driven, but even in the void the impulse would always be diminished in geometric proportion.

For those things that are diminished by two or more causes in geometric proportion are diminished by all these things as by a unique cause that would diminish them in geometric proportion, and the calculation always gives the same result.

By the same token, if another cause retains certain bodies through an arithmetical force, there will always result a diminishing in geometric proportion.

For if another force gives an impulse, acting always in geometric proportion simultaneous to the force that is geometrically diminished, we can finally conclude that the Geometric ends and the Arithmetical remains and augments the movement, as we said the spirit would do in the void (note from Leibniz: thus the force of the spirit in the void is arithmetical).

And, if in the end the impulse increases geometrically and if the movement is still diminished or if it increases arithmetically, the speed increases infinitely in a composite proportion that can be explained by the space made up of the triangle and the curved line (proportionalium) taken together in this manner (fig.1) through addition or (fig.2) through subtraction, in such a way that the speed of the first time is to the speed of the second, as space *abc* is to space *aoed*.

The Latin manuscript of this passage is by Leibniz.' According to Foucher de Careil, it would seem that these are notes to his *Principia Philosophiae* that Descartes appears to have written himself.

It is a question of a note from private study, not communicated to any correspondents. It goes without saying that, far from decreasing the value of these notes, their private nature reveals what Descartes was thinking about and working on in the secrecy of his office. However, the status of this text as an *exercise* undoubtedly allows its author to adopt certain hypotheses and to pursue theories that would be difficult to integrate into his principals of physics.

It is important not to lose sight of the fact that this text is part of a corpus made up of numerous Cartesian texts directly concerning falling objects. We will not discuss the entirety of these writings here. When he intends to take into consideration the real conditions of the fall, Descartes admits, as a general rule, that it is not possible to discover a law of movement, that is, a continuous relationship between space and time. This does not keep him from attempting, with great consistency, to “make an accurate account”of the behavior of a falling object left to itself. It is thus necessary for him to introduce parameters such as the resistance of the environment (which he believes to be proportional to the speed of the moving object) or even the impulse of the subtle substance covering the body in question during the course of its fall (an impulse that varies according to the same speed).

Descartes analyzes the formation of speeds according to a discreet procedure by considering them in the succession of *momenta motus*.

The present text takes up all of these difficulties and seems, at first glance, not to yield any appreciable result. The opposite is true, however, since, in a series of seven propositions, Descartes brings together these diverse fictive modes in order to consider a falling object, from its animation by “a spirit”or “a soul”acting evenly (également) up to its determination by an ensemble - undefined, but infinitely complex - of causes and forces acting together. I plan to show that a common mathematical form addresses all of these situations and, in so doing, to restore the interest and importance of this passage that has gone unnoticed (unneuotised) for too long.

Textual Analysis

Descartes reflects on a series of seven situations here, each one of which is characterized by a certain number of specific conditions, or submitted to certain specific hypotheses, that determine the movement of a falling body. The information that pertains to the first situations clearly establishes the framework of the study: the question of the fall under the effect of weight. We will see that the degree of physical and phenomenal possibility of the established conditions varies from one to the other of the situations.

It seemed possible to me to construct a unique -and de facto coherent- mathematical model for interpreting the seven cases envisioned by Descartes. There is no anachronism here since the resources put to work are by no means outside the ordinary realm of possibilities accessible to an algebraist. If the fundamental architecture of this text has remained implicit, even hidden by its author, it is, I believe, because the mathematical forms in question are not likely to produce measurements, predictions, or quantifiable results that one might confirm through experimentation. Certain parameters cannot be attributed, like the speed of the subtle substance or the coefficients of activity of the causes invoked (resistance, impulse. . .). Add to this the fact that this mathematical model seems to be the only one likely to furnish an interpretive framework for this text.

This sole and unique mathematical means is, in effect, required in order to interpret the entirety of the passage. We have just to know how a recurrent linear sequence becomes explicit. That is, $u_n = au_{n-1} + b$ and $a > 0$; we use the result according to which: $u_n = a^{n-1}u_1 + b \cdot [(1 - a^{n-1})/(1 - a)]$.

If, today, we demonstrate this by a change such as $v_n = u_n + b/(1-a)$, or by recurrence, it is clear that in 1635 one could easily follow the following path:

$u_1 = u_1$ given
$u_2 = au_1 + b$
$u_3 = au_2 + b = a^2u_1 + ab + b$
$u_n = a^{n-1}u_1 + b(1 + a + a^2 + \cdots a^{n-2}) = a^{n-1}u_1 + b \cdot [(1 - a^{n-1})/(1 - a)]$. cqfd

There is, therefore, no problem or obstacle to summoning this result here in order to make this difficult passage from *Anatomica* clear.

The principal behind this interpretation resides in the following idea: as in all texts on the subject, since the years 1618-1619, the speed v_n, at the n$^{\text{th}}$ moment is formed by the addition of the acquired speed v_{n-1} and the new impulse (which we can call I_n) that is potentially corrected by resistance. There is indeed a rule of recurrent formation of speeds at successive stages. It is because we consider the analysis of the movement, first at the level of the *momenta*, that the impulses, speeds, and resistances are rendered homogenous and are susceptible to addition, to interrelation, and can be affected by common coefficients.

Depending on the cases, we will have to consider the following dimensions:
The first impulse given by the subtle substance on the body whose fall we are studying, at the $firstmoment$ of its movement. I will call it i. This dimension i will, without any inconvenience, be taken for the speed at the $first\ moment$, since nothing else enters into its constitution. We can consider $i = v_1$.

The coefficient of diminishment of the impulse with respect to the speed, which I will call k. We can observe that -under certain restrictions concerning the determination- and in calling the speed of the subtle substance V, we can accept that $i = k.V$. It is clearly necessary to note the absence of a vectorial concept of speed since the direction of the movement of the fall is normal vis-à-vis that of the whirlwind of subtle substance. Let's consider the fact that the impulse is a direct function of the speed of the subtle substance, since it determines (di'te:mines) the centrifugal force and the *pressure* that, from this point on, exercises itself on the falling object. The reason for the resistance to the movement which, depending on the case, can be arithmetical or geometric, will be named r. The coefficient (introduced in case n°6) characterizing the action of a force that would augment the impulses will be named l.

In these conditions, an examination of the seven cases envisioned by Descartes, a division that emerges naturally from the text, presents itself in the following manner.

1st case: Descartes evokes the situation of the movement of a body in the void moved by an evenly applied force. We would have, in such a case, a speed that increases arithmetically and we find the habitual Cartesian proportion according to which the time taken to cover the first half of the distance is triple the time necessary to traverse the second half. We will note that such a movement can only be produced a *mente* and cannot be that of a real freefalling body; however, this is a revisiting of results obtained during discussions (with Beeckman) on the fall of bodies[1] (fig.3). At this point I would like to make two remarks:

First remark: By habitual proportion I mean the result adopted in 1619, in texts that emerged from discussions with Beeckman, sometimes referred to as the $proportionoffourthirds$. This result is considered valid in the case of the fall into the void without resistance from the environment. It is employed in the texts of 1619, 1620, then, much later, in 1629,1630, 1631, and 1632. The abandoning of this *proportion* comes much later, after the passage we are reading here. This way this result is obtained will be explicit in the second remark.

Second remark. The notion of speed, at work throughout this extract, is pre-Classical, a qualification that demands some explanation (here I will follow recent comments made by Pierre Souffrin, and I'll take this opportunity to pay homage to this man, a friend, who passed on four month ago).[2] These conclusions can be summarized as follows:

[1] Cf. notably A.T.X, p.75-78; A.T.X, p.58-61, and A.T.X, p.219-223.

[2] We will consult, in particular, P. Souffrin's "Sur l'histoire du concept de vitesse,"in *Le temps, sa mesure et sa perception au moyen-âge,* Paradigne, Caen, 1992.

The term speed, *velocitas* or *celeritas*, used on their own -until at least Galileo- do not indicate the speed at a particular instant, or at "a single point of the trajectory." Then, the notions that anticipate our concept of instantaneous speed were explicit; it is more a question of "the degree of speed," of *gradus velocitatis*, etc. in the terminology coined by Scholastics of the XIVth century, and this can also be the *intensio motus* or even the *impetus*.

The average speed, as a functional relationship between the distance traveled at the time of the trajectory has no value. Speed, as the term is used in the pre-Classical (before Galileo) tradition, and which Pierre Souffrin names "holistic speed" is the measurement of a finished movement, that is, in a period of time already gone by and/or in a space already traversed. As a consequence, whenever it is question of comparing speeds, two readings (and only two) are possible. In equal amounts of identical time, speeds are like the spaces traversed (which does not necessarily bring about uniformity) or in two equal spaces, the speeds are inversely like the times. Another consequence resides in the fact that, when it is a question of the "force of movement," it is a question of its "dimensions," that is, its measurements, and, therefore, of its speed: a "strong" movement is a quick movement. We will find, therefore, a cinematic sense of the term *vis* that, according to the context, can validly be translated as speed.

Admitting this simple proposition, that pre-Classical speed is Cartesian speed[3] and that it designates the measurement of a completed movement, has two consequences:

1. Anything that designates the measurement of the movement can reasonably be considered as a synonym for speed: for example, the "force of movement," the "quantity of movement" (before this expression receives another meaning in the *Monde* and the *Principes*).

2. The figurative representations (generally, triangles and trapezoids) are, consequently, normal representations of that speed; and so, when the spaces traveled are *in extentio*, these figurative representations yield the inverse proportions of the times of the trajectory over equal spaces.

[3] Descartes will radically evolve in his conception of this very important point. This will become manifest in the "later" texts concerning falling objects.

Again in 1643, Descartes writes to Mersenne that "impression & movement & speed, considered in a single body, are all the same thing."[4] Important passages (among which the passage we're considering here can be counted), rendered incomprehensible by the use of Classical concepts of speed, thus pass from obscurity into the light. Indeed, this is what Damerow and his colleagues express on the subject of a well-known passage from the *Cogitationes privatae*[5] "What looks like hopeless confusion if the concept of velocity of classical mechanics is presupposed is hence perfectly reasonable in the logical framework of the concepts involved."[6]

In the implicit schema of this first case, the speeds are like the surface areas of a triangle to a trapezoid, corresponding to the two halves of the trajectory of the fall.

The speed on the part of the triangle being considered results from the accumulation of impulses (the first ones that "continue to act"and those -equal- that are added at each *momentm*).[7] This first case takes on an exceptional character due to the fact that it is susceptible to a smoothing out, by diminishment of the minima up to the point. In the course of a preliminary analysis, the speeds follow the rule of formation already encountered: $v_n = v_{n-1} + i$, more explicitly: $v_n = ni$.

2^nd^ case: The force acting evenly is presupposed once again but it acts in a resistant environment. Descartes affirms that this resistance increases in proportion to the speed of the movement. We are not yet in the situation of the falling body as Descartes conceives of it because, precisely, of the hypothesis regarding the evenness of the acting force. A model for the evaluation of such a situation exists already, however: here we must refer to the letter to Mersenne of December 18, 1629[8] where he describes this point in detail in order to affirm (contrary to Beeckman) that the speed approaches, without

[4] March 23, 1643 to Mersenne, A.T.III, 636.

[5] A.T.X, p.219-220

[6] Damerow, etal., p.31

[7] The process that allows for this accumulation of impulses is detailed in the texts from 1618 (A.T.X, 75-78 and A.T.X, 58-61, and **161ç**-1621 (A.T.X, 219-223). It would be too lengthy a project to give a commentary here, but such a commentary will be included in a forthcoming work to be published on the entirety of these documents.

[8] A.T.I, 92-93

attaining, a maximum value. In the example he discusses, the speeds took the values $\frac{1}{2}, \frac{3}{4}, 7/8, 15/16, 31/32$, etc. at *successive moments*. If one conceives of a series V_n formed from *speeds* to the *n minima motus,* it conforms to the Cartesian declarations that V_n is a convergent geometric series.

In the first moment, one would have an *impulse-speed*, which one might count as 1, from which a proportional resistance must be subtracted. By adopting the value $\frac{1}{2}$ as a base, as in the example of 1629, one obtains $1 - \frac{1}{2} - \frac{1}{2}$ in the first *moment*.

In the second moment, this impulse maintains itself, and another identical impulse must be added to the first (which would make a speed equal to $1 + 1/2$); but a resistance proportional to the speed of the movement must be subtracted. The result is: $1 + 1/2 - 1/2(1 + 1/2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

In the third moment, we have then: $1 + 1/2 + \frac{1}{4} - \frac{1}{2}(1 + 1/2 + \frac{1}{4}) = 1/2 + 1/4 + 1/8 = 7/4$.

In the fourth, will we have: $\frac{1}{2}(1+1/2+\frac{1}{4}+1/8) = \frac{1}{2}+\frac{1}{4}+1/8+1/16 = 15/16$.

These are indeed the values given in 1629, and the speeds increase like a geometric series converging toward 1.

Generally speaking, and in giving an arbitrary value $r (r < 1)$ to the coefficient of proportionality of resistance, one can reconstitute the calculation using our model.[9] At each n$^{\text{th}}$ moment, the impulse is even, equal to I, and the resistance is proportional to the acquired speed, that is, of the form $r.v_{n-1}$ where $0 < r < 1$.

We have then: $I_n = i - r.v_{n-1}$ from which

$v_n = v_{n-1} + i - r \cdot v_{n-1} = (1 - r) \cdot v_{n-1} + I$ (our model of linear recurrence is attained)

v_n can be explained, then, as $v_n = (1 - r)^{n-1} i + i/r[1 - (1 - r)^{n-1}]$

As $0 < 1 - r < 1$, the speed converges toward a maximum value (designated

[9]A. Gabbey dedicated several important pages of his thesis to such an effort; I have indicated how my interpretation differs from his (cf. Gabbey thesis, p 320sq).

here as i/r); a result that is in accordance with the conclusions about the "point of equality"evoked in 1629 and with Descartes' renewed assertion according to which "we ultimately arrive at a point where speed is no longer perceptibly augmented and it becomes possible to determine a certain other final speed to which it will never be the equivalent."
We quickly have, then, a quasi uniform movement, which Descartes emphatically defends.

It is not possible, however, to establish from this - without considerations other than those taken into account by Descartes - a law of movement or even a relationship between the doubling of distances traveled and the time necessary for this doubling.

3rd case: The fall is not really imagined in a void but, more generally, in circumstances where speed increases. This is the key point of this passage: as soon as speed increases, the impulse given to the falling object is no longer even. What is at stake here is the doctrine of the gravity-pressure caused by the subtle substance (even if this matter is not explicitly designated), which varies according to the speed of the falling body. It is, therefore, accurate to state that in all cases where speed increases (and, then, of course in the hypothetical case where there is no environmental resistance) the impulses decrease. The "secondary"hypothesis to accept is that this decrease is geometrically proportional. We will observe that Descartes does not draw any conclusions concerning the speed - and thus the nature - of movement using this consideration alone. This important case draws - for the fall - the necessary conclusion according to which the effect of an invariable cause on an object whose state has been modified cannot itself be invariable. In effect, it is possible to successfully apply our model by exploiting the Cartesian propositions concerning the sequence of impulses. Let us consider the fact that the subtle substance is animated by a speed V. It gives a first impulse i in the direction of the fall, which produces the speed v_1 (or I) in the *first moment.*

In each n$^{\text{th}}$ moment, the subtle substance gives an impulse that diminishes in function of the acquired speed. What counts, in fact, is the difference between the speed of the subtle substance and the speed acquired by the object. We must then consider that the impulse is proportional to this difference, like $k \cdot (V - v_{n-1})$ where $0 < k < 1$. This is corresponds to the Cartesian assertion that "even in the void, the impulse would always be diminished in geometric proportion."

The general rule of speed formation is thus:

$v_n = v_{n-1} + k \cdot (V - v_{n-1})$ or $v_n = (1-k) \cdot v_{n-1} + k \cdot V$ or $v_n = (1-k) \cdot v_{n-1} + i$ (since $k \cdot V = i$)

The model of linear recurrence is attained once again. We notice its striking resemblance to the preceding case. The resistance proportional to the speed plays exactly the same role as the impulse proportional to the difference $(V - v_{n-1})$.

v_n can thus be explained as $v_n = (1-k)^{n-1} i + i/k[1 - (1-k)^{n-1}]$

As $0 < 1 - k < 1$, the speeds converge toward a maximum value (designated here as i/k). It is not uninteresting to interpret this maximum value i/k, taking into account that $i = k.V$. We would then see the speed converge toward V, which represents the speed of the subtle substance.

4$^{\text{th}}$ case: This passage is based upon the results of the preceding analysis. In order to discover how speeds form under the double influence of resistance and variable impulse, it suffices to be faithful to the general schema of the formation of movement and to have the two influences act in concert. We will make this first impulse i, and make k the coefficient of variation of the new impulses at each *momentum*. For a speed v_n, the new speed is not i, but $k \cdot (V - v_{n-1})$. We will call r the coefficient of resistance proportional to the speed. A force like $r \cdot v_{n-1}$ enters, then, into the constitution of v_n.

In each **n$^{\text{th}}$** *momentum*, there is the following recurring relationship between v_n and v_{n-1}.

$v_n = v_{n-1} + k \cdot (V - v_{n-1}) - r \cdot v_{n-1}$ where $0 < k < 1$. or $v_n = (1-k-r) \cdot v_{n-1} + k \cdot V$ or $v_n = (1-k-r) \cdot v_{n-1} + i$ (since $k \cdot V = i$)

Again, we can use the same linear model.

v_n can thus be understood as $v_n = (1-k-r)^{n-1} i + i/(k+r)[1 - (1-k-r)^{n-1}]$ and v_n converges toward $i/(k-r)$

The "calculation"evoked by Descartes is indeed of the same sort as the preceding ones; it "always yields the same thing."We notice that the speed itself (and not only the impulses) can decrease if $k + r > 1$. This only makes sense if we adopt the doctrine of the non zero-value immediately acquired by the initial speed.

5th case: This passage, like the following ones, is quite difficult to interpret. If we contend that Descartes focuses here on the phenomenon of the fall, we identify the first cause, which expresses itself as a geometric proportion, with the variable gravity produced at each *moment*, an augmentation that corresponds to a convergent geometric progression. But it is perhaps not very pertinent to insist on furnishing a foundation at the end of this text that is based in reality. This is confirmed by the characterization of the second cause, the one that retains or resists, as an arithmetical force, unintelligible if we stick to a description of actual phenomena.

We can still interpret things according to our same model, with i as the speed at the *first moment*. The impulses decrease proportionally to the difference between V and the acquired speed v_{n-1} (as $k(V - v_{n-1})$, where $0 < k < 1$); but, at each moment a cause, estimated as r, retains the moving body. The rule for the formation of speed at the n^{th} moment can thus be expressed as:

$$v_n = v_{n-1} + k \cdot (V - v_{n-1}) - r \quad \text{or} \quad v_n = (1 - k) \cdot v_{n-1} + k \cdot V - r \quad \text{or}$$
$$v_n = (1 - k)v_{n-1} + i - r$$

A formula that once again corresponds to linear recurrence. We can, then, also deduce that:

$v_n - v_{n-1} = (1 - k)(v_{n-1} - v_{n-2})$. "A diminishment in geometric proportion will always result,"comments Descartes.

This series can be explained as $v_n = (1-k)^{n-1} \cdot i + (i-r)/k \cdot [1-(1-k)^{n-1}]$.

As $0 < 1 - k < 1$, v_n converges toward $(i - r)/k$ (or $V - r/k$)

We notice here, as Descartes clearly indicates, that the impulses and the resistances are homogenous in their dimensions and can be added to and subtracted from on another as such.

6th case: We recognize two causes acting here, as if in opposition, according to geometric progressions (one to increase the impulses, the other to diminish them). If the second cause is almost familiar to us, it is clear that the first is a pure fiction, or an artifact difficult to conceive of, a sort of supplementary motor, no example of which can be found (except perhaps in the **aeroliypl**?)

Our algorhythm, henceforth *standard*, offers us a satisfactory interpretive frame. We can adopt the following formula as the rule of formation for successive impulses: that is, a first cause that gives geometrically diminished impulses, of the form $k \cdot (V - v_{n-1})$ to the n$^{\text{th}}$ moment (in accordance with that which has been established in the preceding cases); that is, yet *another force* that acts *simultaneously* in proportion to the acquired speed according to a coefficient l. This second impulse is thus of the type $l \cdot v_{n-1}$. The total impulse at the n$^{\text{th}}$ moment is thus:

$$I_n = k \cdot (V - v_{n-1}) + l \cdot v_{n-1} = kV + (l-k)v_{n-1} \text{ or } I_n = i + (l-k)v_{n-1}$$

Consequently, it is not impossible to determine an arithmetic progression of speeds, that is, an impulse acting *like a soul*. It suffices to take $l = k$, which is compatible with the Cartesian claim. We may simply accept that the cause that increases the impulses (proportional to $(V - v_{n-1})$) has the same intensity as that which diminishes them (proportional to v_{n-1}), as in a effect of symmetry. In this case, where the coefficients of the impulses, amplifying and attenuating respectively, are equal, we have: $I_n = kV = i$.

"We finally arrive at the point where the Geometric ends and only the Arithmetical remains,"writes Descartes. We have reestablished exactly the *ideal* case number 1, where only a "soul"acts.
As I have said already, the phenomenal interpretation of such a formal expression is completely improbable.

Otherwise, where $k\#l$, we have
$v_n = v_{n-1} + (l-k) \cdot v_{n-1} + i = (1 + l - k) \cdot v_{n-1} + i$ (linear recurrence once again) $v_n = (1-k-l)^{n-1} \cdot i + i/(k-l) \cdot [1 - (1 + l - k)^{n-1}]$.

7$^{\text{th}}$ case: It is clear that from here on - at the least - we enter into perfectly imaginary considerations. It would seem that no simple circumstance of the actual world might produce falls in the course of which impulses would increase. The rather carefully crafted figures added by Descartes nevertheless allow us to appreciate what he was thinking about here. Their general organization is of the same type as that of the *triangular* figures in preceding texts. The surface areas formed from the line of the *extensio* onward are the measurements of the movement, obtained by the accumulation of impulses, considered on the *momenta* without width (largeur) (since the figures are "smoothed out"). In the two figures, two sorts of impulses (which generate the speeds) are combined at each moment. The curved lines correspond to geometrically increasing impulses; they thus must be considered as fictional since the real conditions of the fall produce concave curves (where the augmentation diminishes). A variation of a arithmetical nature is associated with them (diminishing or augmenting the impulses). It is the two triangles - Classical triangles - that represent this variation. The measurement of the movement is thus naturally yielded by the addition or the subtraction of the surface areas. In the two cases, the combination of the two produces an infinite progression of the speed.

The figures allow for the conception of a law of movement, since they express the proportion of the speeds (in the pre-Classical, or global, sense, if we follow the figures) in function of the trajectory. A noteworthy difference must nonetheless be indicated, since the relation of the speeds is evoked according the time passed, "primi temporis,"and not according to the spaces. This relation is, evidently, not calculated.

Once again, it is not impossible to come up with calculations that reestablish the coherence of the Cartesian results. I will consider here, then, a speed at the first moment represented by v. The impulses presupposed as being in increasing geometric progression, without further precision, we will be able to attribute a certain value i to the first of them (which in the passage also designates the first speed $i = v$) and designate the series of successive impulses as $(k^n i)_{1<n}$. Lastly, at each moment, a resistance r arithmetically diminishes or augments the movement. The speed vn is linked to the preceding concept by the relationship:

$v_n = v_{n-1} + k^{n-1} \cdot i + r$. This relationship is the addition of our "standard form"and a simple geometric progression. It can be explained as

$$v_n = i + (n-1)r + k \cdot i[1 - k^n/1 - k].$$

It is rather satisfying to notice that this formula corresponds well to the schemas that Descartes offers. The portion of the formula "$i + (n-1)r$"corresponds to the triangles and thus to the arithmetical augmentation, while the portion of the formula "$k \cdot i[1-k^n/1-k]$"corresponds to the part defined by the curve. "The speed increases infinitely in composite proportion, which can be explained by the space made up of the triangle and the curved line taken together."Depending on whether r is positive or negative, the two movements are added or subtracted and, in the two cases, as $k > 1$, the speed *at the moment n* "increases infinitely"if n becomes infinitely large.

General remarks

This text, *in isolation*, is in some ways the pinnacle of the Cartesian attempt to produce a mathematical expression of the fall of bodies. There is a *before* and an *after* this passage of the *Anatomica*. Availing of his theory of elements, of mechanism, and of weight, Descartes tries to render them all coherent vis-à-vis the proportions that would describe their effects during the fall. This takes into account the *increasing complexity* of the question up to the seventh case.

1. From the point where Descartes no longer makes use of the temporal function onward, and where he systematizes a model of impulses to present the diverse cases, Descartes is no longer in a position to deduce a relationship between the times and the spaces based on the consideration of speeds at successive *moments* of the movement. Because of its uniform nature, the first case remains unaffected by this difficulty. In effect, if, as in the texts of 1619-1620, we pursued the thought process by *diminishing the moments* until we arrived at the *veritable minimum*, that is the point, we would obtain a situation where the speed would tend immediately toward its maximum value in all cases where it converges (that is, cases two, three, four, and five). As such, the movement of the fall would be almost immediately uniform or quasi uniform. We should connect this remark with a passage from the letter to Mersenne from December 1629 where he wrote:

   "But returning to the falling weight, one can see by this calculation that the unevenness of the speed is very significant at the outset, but almost imperceptible afterward[...] For, according to this calculation, and taking only a very small space for an moment, one will find that a ball that falls 50 feet goes almost three times more quickly at the second inch than it did at the first and, however, that at the third foot it does not go perceptibly more quickly than at the second and will not take any more time to fall the first 25 feet than the last 25, aside from what it takes to fall two or three inches, which would be completely imperceptible.[10] As Descartes writes, *according to this calculation*, if we took even a point for an instant, the fall would be immediately uniform. His math didn't allow him to deduce a law of motion from his hypothesis of speeds as the sum of a geometric series. The absence of the temporal variable and methods of integration of functions produces its effects.

2. The quest for a mathematical model for the diverse cases envisioned cannot be crowned with success unless one seriously considers the precise distinctions made by Descartes between the impulses and the speeds. It is indeed the former that immediately take into account the causes and the forces that condition the fall. The calculation of speeds is itself a consequence of the first level of formalization, inevitable consequence once we admit the constitutive schema of speeds in each moment, as composites of the acquired speed and new elementary forces at work (impulses and resistances).

3. In the analysis of the third case, I chose to posit that the impulse had to be proportional to the difference $(V - v_n)$ This choice offers the considerable advantage of reestablishing the general coherence of the diverse cases that follow. It so happens that we are further authorized to do so by an attentive reading of the letter to Mersenne of March 11, 1640, where it appears clearly that the determination of the variation of speeds is a direct function of the difference that we have made manifest.

4. We must further observe that, if the discussion of the conditions of the fall makes use of the principles of physics (impossibility of the void then variable gravity), the hypotheses that should allow us to give the proportions of the phenomena are not deduced from these principles. The ratio that characterizes the resistance of the environment and the ratio that characterizes the variation of gravity, in the diverse cases envisioned, are not given. In the text of 1629, the first ratio (which was the only one discussed) is relatively arbitrary (it might be $\frac{1}{2}$, or 4/5, we only know that it is less than 1). Thus one might put forth the idea

[10] A.T.I, p.94-95

that, submitted to the to the full Cartesian doctrine and to Cartesian gravity, the nature of the fall is mathematically elucidated: that speed increases, like a geometric series toward a maximum value, but that it cannot be calculated, since the quantification of the hypotheses retained is not deducible by the laws of nature.

5. What might possibly be the status of such a study? The objective, the subject itself, is particularly ambitious and comes from what Descartes had called a *veritable science of motion*.[11] That which is aimed for here is a mathematization of the fall, understood as a complex phenomenon, dependant at once on the laws of nature and also on more or less arbitrary hypotheses. The end result is a nuanced total: if the speed is the object of a general mathematic expression, it stands that no relationship between times and spaces can apparently be deduced.

6. We must further take into account the possibility, henceforth established, of a coherent interpretation -on a mathematical level- of the seven propositions. Strict adherence to the process of formation of the *minima motus* in successive *momenta* permits us to quantify, in effect, or more exactly, to form an algebraic expression corresponding to the imagined conditions. However, two sorts of interdictions radically dissociate these implicit equations from these phenomena: the first concerns the impossibility of interpreting *in reality* the causes at play. What might well be the force that would produce a geometrically increasing impulse, or that would give rise to an arithmetical resistance? The second hangs on the fact that the parameters that must necessarily be introduced into the equations are unassignable, like the speed of the subtle substance, its first impulse, the coefficients of proportionality employed, etc.

   Such are the reasons that explain why, despite the extreme elegance of this text and the rigor of its organization, it remains allusive and emblematic of the Cartesian decision to not push any further his attempts to furnish a quantifiable law of motion, in the real, non-negligeable conditions of the world.

   About ten years later, Descartes will come to the Galilean result and will produce a very good explanation of the Galilean law of falling

[11] A.T.X, p.223

bodies, but he never pays tribute to the Italian and maintains the result is only true in very specific situations in witch on can forget resistance of the air and gravity variation.