LEARNING TO PROVE: ENCULTURATION OR…?

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Empirical evidence coming from a curriculum innovation experience that we have been implementing in the Universidad Pedagógica Nacional (Colombia), in a plane geometry course for secondary mathematics pre-service teachers, allows us to affirm that learning to prove, more than enculturation into mathematicians' practices, is participation in proving activity within the community of mathematical discourse.

AN EXPERIENCE FOCUSED ON LEARNING TO PROVE

Our contribution is linked to the curriculum innovation experience that we have been implementing since 2004 in the framework of a pre-service program, for high school mathematics teachers, that includes a high percentage of credits in mathematics formation. The experience takes place in an 80 hour plane geometry course, 2nd of the 6 courses in the area of geometry. Students' ages are between 18 and 21. Since upon entering the University, their geometry content knowledge, and know-how, and their mathematical argumentation experience are minimal, in their first geometry course the approach to geometric objects is informal. The intention is to provide experiences that help students construct or amplify their geometric background, and improve their disposition and preparation for commitment in the next course.

The plane geometry course's goal is to create opportunities to learn to prove that should affect students' conception of proof not only from a mathematical point of view but also from a didactical one. Besides learning to build deductive chains, we expect students to recognize the role of proofs as a resource for understanding and arguing and as a fundamental activity in mathematics tasks.

As we pursue the course's purpose, we defy traditional university mathematics teaching practice: we embark in a **collective construction** of an axiomatic system related to points, lines and planes, angles, properties of triangles and quadrilaterals. To achieve this enterprise, teacher and students participate jointly in mathematical activity, articulating practices such as defining geometric objects, empirically exploring problems, formulating and verifying conjectures, and writing deductive arguments. It is through the questions and tasks proposed to the students by the teacher, that the course is developed; that is, the geometric content treated in the class doesn't originate from a textbook, nor is it presented by the teacher. This situation can appear unusual and surprising if one asks how students can participate in the creation of mathematical discourse, unknown to them; it is possible due, principally, to three reasons: the instrumental mediation of a dynamic geometry program (Cabri), a clearly delimited reference framework, and the teacher's management of the class, coherent with the course's purpose.

DAWNING OF PROVING ACTIVITY: THREE PRACTICES

We are aware that a practice entails not only the actions through which it materializes but also repertories, work routines, values, interests, resources for negotiating meanings, etc. (Wenger, 1998). Even so, due to lack of space, we sketch below three mathematical practices, focusing primarily on the actions.

Analyzing a definition. When a term appears, in a question or task set by the teacher or in a student answer, which will be part of the specialized vocabulary, including it in the axiomatic system requires a precise definition that will be elaborated jointly by all the class members. They are terms that the students have an intuitive idea about and, therefore, can make a graphic representation and verbalize a statement that becomes the first version of the definition. Whether it coincides or not with the definition that will be institutionalized, the teacher leads a process which includes examining the coherence between the given verbal statement and its graphic representation, graphically presenting cases that should be excluded from the definition and are being included or vice versa. The analysis appeals to questions like “What if such a condition is excluded?”, “Why is it required?”, “Do these statements define the same object?”, or specific questions related to the object itself. For example, after a student’s definition for *segment*: “The set formed by points A, B and those between A and B”, was accepted by his classmates, the teacher focused their attention to the characteristics of this geometric object with questions like: “Is $\overline{AB}$ a subset of some line?”, “Which one?”, “How do we know?”. Answering the questions involved the class community in the collective production of a proof, product of the following considerations: (i) $\overline{AB}$ has more than two points and therefore the inclusion of $\overline{AB}$ in $\overleftrightarrow{AB}$ can not be justified by alluding to the fact that A and B belong to the segment and to the line; since $\overline{AB}$ has at least a point C different from A and B, it is indispensable to show that this point is also an element of the line through A and B; (ii) points A, B and C of $\overline{AB}$ are collinear since betweeness, which characterizes a segment, includes this condition; (iii) the line that contains A, B and C is the same one determined by A and B because two points determine a unique line. That proving activity took place in the 6th class through a conversation guided by the teacher, conformed in all by 150 interventions, 70 by the teacher and 80 of 12 of the 21 students that constituted the group.

Enunciating propositions. Some of the propositions proved in the course guarantee the existence of a geometric object. Initially, they are conjectures, suggested by the students as answers to a problem, worked on in small groups with a dynamic geometry program, exploring possible constructions of the object whose existence must be proven. However, students’ answers usually are not expressed as a conditional statement or, if so, a condition that should be part of the antecedent is not included, or antecedent and consequent are interchanged. A public revision of the statements to determine whether it must be reformulated is essential. Given Cabri’s mediation in the process, the revision is centered on determining whether correspondence exists between what was done with Cabri and what the conjecture states, bringing out the given conditions and determining which consequences result.

The following episode, which took place in the 21st session of the course, illustrates characteristics of the above practice. As response to the given problem, Group 1 and Group 2 formulated their conjectures.

Problem	**Conjectures formulated**
Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be opposite rays and $\overrightarrow{AD}$ another ray. Is it possible to determine a point E, in the same half plane in which D is found, for which $\angle BAD$ is complementary to $\angle CAE$?	If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays and having $\overrightarrow{AD}$, then there exists a point E, in the same half plane in which D is found, such that $m\angle EAD = 90$ and $\angle EAC$ and $\angle DAB$ are complementary. [Group 1]
	If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, $\angle BAD$ is acute, $E \notin \text{int}\angle BAD$ and $m\angle DAE = 90$, then $\angle CAE$ y $\angle BAD$ are complementary. [Group 2]

In their report, Group 1 presented the details of the Cabri construction and exploration carried out; the group constructed the figure as required: they constructed $\angle EAC$ complementary to $\angle DAB$, dragged to vary their amplitude, noticed that the amplitude of $\angle EAD$ remained invariant, measured and found it to be 90°. The correspondence construction-conjecture was studied through teacher questions which lead students to focus on specific aspects that aided determining whether the correspondence existed or not; she asked questions like “Did the group construct rays $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ as required by the problem and just with that noticed that an angle was right and all the rest?”, “Besides constructing the three rays, did they construct a fourth ray that satisfied a certain condition?”, and “Was the construction of $\overrightarrow{AE}$ carried out to construct $\angle EAC$ or $\angle EAD$?”. When the answers to these questions were discussed, students realized that the group had constructed $\angle CAE$ to be complementary to $\angle BAD$ and therefore obtained the existence of $\angle EAD$ which turned out to be a right angle. The analysis showed that the Group 1’s conjecture didn’t coincide with the construction and information extracted. Reformulating it was necessary, since the class was convinced of the regularity evidenced by the empiric experience.

Having the reformulated statement, its relation with the conjecture given by Group 2 was examined. For this, the teacher posed the following question to the other members of the class: “Suppose correspondence between construction and conjecture exists. How do you imagine the construction process was?”. In the analysis, hypothesis and conclusion were identified, thereof explicitly setting the four conditions, included in the antecedent of the conditional that must have been constructed to obtain what the consequent expresses. It was then realized that second conjecture was almost the reciprocal of the reformulated first one.

Submitting a proof for consideration. The students start to participate actively in the construction and evaluation of proofs from the beginning of the course, as the following example illustrates. Students were asked to prove: *Given a line and a point not on it, there is exactly one plane containing both of them*. After allotting time for students to reformulate the statement as a conditional and devise a plan for the proof, Ana, voluntarily, writes on the board the proof she and Juan produced; students were asked to be vigilant so as to approve it or not Ana's proposal. She reformulated the statement as: "If $\overleftrightarrow{AB}$ exists, and a point F, that doesn't belong to $\overleftrightarrow{AB}$, then there exists a plane α such that $\overleftrightarrow{AB}$ union F is contained in α." Juan immediately intervened to point out that the plane is unique. Ana wrote the steps of the argument as she verbalized it, as shown:

	Statement	**Justification and steps involved**	
1	$\overleftrightarrow{AB}$ exists.	Given.	
2	Point F exists that doesn't belong to $\overleftrightarrow{AB}$.	Given.	
3	Points A and B exist that belong to $\overleftrightarrow{AB}$.	Line theorem. (Every line has at least two points.)	1
4	A, B, F are non-collinear.		2, 3
5	There exists exactly one plane α.	Plane postulate. (Three non-collinear points determine exactly one plane.)	4

Once finished, Germán objects on how Ana mentions the line in her first step because there she was "assuring the existence of the two points… if I declare them from the beginning, I am giving the existence of those points". He proposed writing this statement as "*m* is a line". Juan intervened, indicating that Ana and Germán were saying the same thing and that the issue was simply one of notation. His counterargument was: "Well, the way I see things, she is giving the existence of the points because when she says 'line *AB*' she is mentioning where the line passes. It is very different to say line *m* because it doesn't indicate where the line passes". Daniel, although expressing agreement with Germán, saw no problem with Ana's notation, and commented that it was possible to write a better statement: "It is better to use *m* or *n*, any name, and then apply the theorem to obtain the two points". The teacher intervened to analyze the situation, indicating that the issue is just notation because the existence of the two points was not taken as given but was deduced from the line's existence; to illustrate the issue more clearly, she said: "If we change the expression 'line *AB*' for *m*, notice that practically nothing changes in the argument; we would only have to change 'line *AB*' for *m*' in step 3". However, she pointed out that "it might be more elegant to express it as Germán and Daniel suggest". Once the discussion was finished, attention was set on what Daniel labeled as a "trivial" point, a step of the proof that was lacking: "I think saying the line is in the plane is missing", which lead to Ignacio´s question "Don't we need to mention the Flatness Postulate (if two points of a line lie in a plane then the line is in the same plane) to say the line is in the plane?", which led to modifying the proof. This episode took place in the 22nd session of the course, through a mathematical dialogue that included 26 interventions, 10 of the teacher and the rest, of 6 students.

DISCUSSION

What do the above examples say about what learning to prove in our course is? Firstly, and although no details of the interaction were included in the sketches, we think that it's possible to envision on it the student participation in the proving activity through which the course is developed; *proving activity* refers to all the actions involved in the formulation of conjectures and the production of deductive justifications based on the axiomatic system constructed by the class community. This characteristic feature can be associated with the idea that *learning to prove* is a process through which students acquire more capability to participate in proving activity in a genuine (i.e., voluntarily assuming their role in achieving the enterprise set in the course), autonomous (i.e., activating their resources to justify their own interventions and to understand those given by other members of the class community), and relevant form (i.e., make related contributions that are useful even if erroneous). Since participation occurs in a community which begins to form as soon as the course begins, and is made up, with respect to learning to prove, by apprentices and only one expert, the teacher, we don't see the process as an enculturation one. We understand that the concept, *enculturation,* has to do principally with a way of knowing, proper of a cultural group and linked to dispositions, forms of acting, beliefs and values that characterize the group: specifically, enculturation is the process through which a person acquires a group's culture due to his interaction with the group's members and observation of their interactions as they carry our their practices. Instead, we see as more appropriate the idea of formation of a *community of practice* (Wenger, 1998).

About the first example, we underline that the conceptualization process gives rise to proofs that justify the answers to questions posed. In those cases, the function of proof is not to validate or verify a conjecture so as to incorporate it in the axiomatic system; instead, it is to help understand the implications that the analyzed statement has, function recognized as important by mathematicians. In the second example, we bring out, not only the type of task that focuses attention on the existence proof of an object through a characterization that permits its construction, but also, the emphasis placed on the comprehension of conditionality and its expression as a statement. Students' expertise with the notation and specialized vocabulary must be remarked on. In the third example, related to the practice of submitting a proof produced by one of its members to the community's criticism, and making relevant criticism, undoubtedly reflects one of the most important practices of mathematicians. Also, we can point out not only the deductive axiomatic character that the arguments presented as proofs possess but also the rigor with which we seek to work, reflected in the issues students pay attention to when they comment a fellow student's production. We consider their preoccupation for controlling the unconscious action of using in the justification that which is being justified very valuable.

The above remarks allow us to argue that the community of practice conformed worries about and undertakes mathematical issues related to proof from a Euclidean geometry point of view. That makes our community of practice fitting in perfectly into what Ben-Zvi and Sfard (2007) consider as a community of mathematical discourse. For them, *discourse* is a type of communication, established historically, that congregates a human group and segregates it from other groups; the membership in the wider community of discourse is achieved through participation in communicational activities of any collective that practices this discourse, no matter what its size is; and to belong to the same discourse community, individuals don't have to face one another and don't need to actually communicate. We think that considering the community of practice conformed in our course as a micro-culture of the community of mathematical discourse expresses the fact that mathematics is present when making curricular decisions. Therefore, although we don't see the process of learning to prove as enculturation into the practices of mathematicians, we are interested in, and think we achieve it to some measure, student acquisition of dispositions characteristic of mathematicians as, for example: (i) preference of the if-then format to express propositions and the use of a particular generic to make deductive reasoning agile; (ii) controlled use of graphic representations, with clearly established conventions, to support the statements that conform the final proof; (iii) careful use of terms and notation for geometric objects; (iv) exclusive recursion to the axiomatic-deductive system for the justifications of the statements in a proof; (v) acceptance of the convenience of a detailed deductive process; (vi) belief that proving activity involves exploring, conjecturing, searching ideas for a justification, producing a proof based exclusively on the theory constructed and submitting the production to criticism.

How are these dispositions developed in the students? Without doubt the type of tasks in which students systematically participate, from the beginning of the course, the collaborative work between students, the teacher's role as expert of the community with whom students interact, the instrumental mediation of dynamic geometry are decisive factors in the formation of such dispositions.

REFERENCES

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