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GENERALIZATION IN FIFTH GRADERS WITHIN A FUNCTIONAL APPROACH

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This article discusses evidence of fifth graders' (10-11 year olds') ability to generalize when solving a linear function problem. Analyzed in the context of the functional approach of early algebra, the findings show that students generalized both when solving specific problems and when asked to define the general formula. The results are described in terms of the type of questions in which students generalized their answers, as well as of the functional relationships identified and the types of representation used to express them. Most of the pupils who generalized did so based on the correspondence between pairs of values in the function at issue.

INTRODUCTION

Research interest is growing around elementary school students' understanding and expression of notions about algebraic concepts (Blanton, 2008). Algebraic thinking plays a key role in research on school algebra, for it entails the development of the ability to analyze relationships between quantities, deduce general patterns and use symbols to represent ideas, among others (Kaput, 2008; Kieran, 2004). Functional thinking is the component of algebraic thinking focused on in this study. In particular, elementary school students' ability to generalize is explored in the functional approach to algebraic thinking. Students may express generalization, the key to such thinking, in words or, given time, symbolically (Blanton, 2008).

Functional thinking addresses the relationship between two (or more) variables: specifically, it involves the types of thinking that range from specific relationships to the generalization of relationships (Smith, 2008). Although such thinking appears to be beneficial for students, its application in in the lower grades has received scant attention (Blanton & Kaput, 2011). In Spain, functional thinking is a fairly recent area of research that has yet to be fully explored, although some of the findings in connection with early schooling have merited international interest (e.g., Cañadas & Morales, 2016). This study was preceded by research on fifth graders' ability to generalize from contextualized problems and the systems of representation used to express such generalization (Merino, Cañadas, & Molina, 2013).

Those studies revealed a need for further exploration of fifth graders' ability to generalize when establishing relationships between variables. In addressing that need,

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this paper focuses on the generalization displayed by such students when solving a problem involving a linear function.

GENERALIZATION AND REPRESENTATION

According to some researchers, generalization, the key element in algebra, is present when students intuitively perceive a certain underlying pattern, even though they are unable to represent it clearly (Mason, Burton, & Stacey, 1988). Generalization implies deliberate reasoning that builds on specific cases to identify inter-model, inter-procedural or inter-structural relationships (Kaput, 1999). Krutestskii (1976) identified two levels of generalization: (a) seeing what is general and known in a specific instance; and (b) seeing something general and still unknown in an isolated instance.

Algebraic symbolism has been directly associated with generalization in different grades. Moreover, other types of representation, including verbal, numerical, pictorial and manipulative, are of interest in the context of early algebra (Kaput, 2008; Merino, et al, 2013). Stacey (1989) distinguishes two kinds of generalization: (a) *near generalization*, for questions "which can be solved by step-by-step drawing or counting", and (b) *far generalization*, for questions "which goes beyond reasonable practical limits of such a step-by-step approach" (p. 150).

In a recent study, Blanton, Brizuela, Gardiner, Sawrey and Newman-Owens (2015) explored lower grade students' ability to generalize in problems involving linear functions. Their findings distinguished between students who identified a specific and those who detected a general relationship between variables, and related the distinction to the ability to symbolize. Students who established the relationship between variables for specific cases "did not yet have a representational means to compress multiple instances into a unitary form that could symbolize these instances" (p. 542).

FUNCTIONAL THINKING

Functional thinking is a "component of algebraic thinking based on construction, description and reasoning with and about functions and their constituents" (Cañadas & Molina, 2016, p. 210) that ranges from specific relationships to generalizing the relationships between two (or more) variables (Smith, 2008). In most countries, students are not introduced to functions, which comprise the core content of this type of thinking, until secondary school. The present study used the linear function $f(x) = ax + b$ (with the domain and codomain limited to natural numbers) as a port of entry for early algebra to afford students the opportunity to explore variations in quantities (Blanton, Levi, Crites & Dougherty, 2011).

The study focused on bivariate functions. Smith (2008) defined the functional relationships involving two quantities that co-vary to be: (a) correspondence, or the relationship between the pairs of values for the two variables (*a, f(a)*); and (b) covariation, or the relationship that describes how changes in one variable affect the other.

METHOD

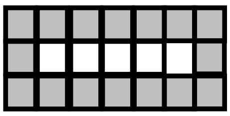
This study forms part of a broader teaching experiment on functional thinking in fifth graders in which the contextualized problem posed in each session revolved around a linear function. This article discusses the results of the fourth and final session, when student progress was greatest because they had already worked on a number of problems involving functions.

Subjects and tools

The 24 subjects were fifth graders (10- to 11-year-old) enrolled in a school in Granada, Spain, who were deliberately chosen on the grounds of school and teacher availability. In the first three sessions of the teaching experiment, the students worked with contextualized problems for which the functions were: $f(x)=2x$, $f(x)=3x$ and $f(x)=3x-7$. The students had not worked on problems involving functions prior to these sessions.

The research team consisted in the teacher-researcher who led the sessions and two researchers who recorded the videos and helped answer students' questions. In the tiles problem posed to all students, the implied function was $f(x)=2x+6$. The problem and related questions are reproduced in Figure 1.

A school wants to re-pave its corridors because they are in poor condition. The faculty decides to use a combination of white and grey tiles, all square and all the same size. They are to be laid as in the drawing.



The school contracts a company to re-pave the corridors on all three floors. We want you to help the workers answer some questions before they get started.

Q1. How many grey tiles will they need for a corridor with 5 white tiles?

Q2. Some corridors are longer than others. So the workers will need a different number of tiles for each corridor. How many grey tiles will they need for a corridor with 8 white tiles?

Q3. How many grey tiles will they need for a corridor with 10 white tiles?

Q4. How many grey tiles will they need for a corridor with 100 white tiles?

Q5. The workers always lay the white tiles first and then the grey tiles. How can they figure out how many grey tiles they need if they have already laid the white ones?

Figure 1: The tiles problem

The questions posed involve: (a) specific instances (Q1, Q2, Q3 and Q4) and (b) the general case.

The information gathered included the session videos and the students' answers to the questionnaire. This article describes the results deduced from the students' written responses.

Analytical categories and data analysis

Category construction was based on grounded theory, which deems that phenomena are not conceived statically (Corbin & Strauss, 1990). The theoretical framework, background and characteristics of the contextualized problem were applied to define some of the categories. Generalization was identified based on its presence or absence in students' replies to the questions, with a focus on the answers where it was detected. Drawing from the ideas on generalization relevant to the conceptual framework of this study, a preliminary analysis of the data revealed two types of questions in which students exhibited generalization: (a) in Q1, Q2, Q3 and Q4, where they were asked to reply to specific (near or far) questions; and (b) in Q5, where they were (directly) asked to generalize. These two types of generalization were respectively labelled *spontaneous* and *prompted* generalization.

Students' generalization was described in terms of the functional relationship generalized (correspondence or covariation) and how it was represented (verbally, with algebraic notation or combinations of one or the other or both with other systems).

Students were labelled as S_i where $i = 1, \dots, 24$.

RESULTS AND DISCUSSION

Of the 24 students, five gave direct answers only (i.e., only the numerical result), described how they counted the tiles or simply repeated the problem: no generalization could be attributed to these pupils. The other 19 answered at least one of the questions in a way that attested to generalization. Two profiles were identified: (a) three students exhibited both spontaneously and prompted generalization; and (b) 16 students generalized only when prompted (when replying to Q5).

Two of the students who generalized spontaneously and when prompted (S5 and S8) used algebraic notation to represent their replies. S8's answer to Q1 was: "formula: $(x\cdot 2) + 6 = 16$; x = number of white tiles." Figure 2 shows how this student related the pairs of values (number of white tiles-number of grey tiles), given five white tiles.

White tiles (x)	Grey tiles (y)
5 —(x·2)+6→	16

Figure 2: Example of correspondence, S8 in Q1

Figure 2 also illustrates S8's use of algebraic symbols "(x·2)+6" to express the general relationship. In Q2, Q3 and Q4, this student simply answered the questions. That was interpreted to mean that the student used the same functional relationship for 8, 10 and 100 tiles, relating the pairs of values (a, f(a)) for a = 8, 10 and 100 and correctly finding that the number of grey tiles needed would be 22, 26 and 206, respectively.

S6, the third student who generalized spontaneously and when prompted, described the generalization in Q1 in the following words: "they need 16 grey tiles. For every white tile, there are 2 grey tiles, except on the sides, where there are 6. All the whites x2 + 6 on the sides". Hence S6 identified the relationship between variables as well as the constant number (six white tiles on the sides). This student used both verbal and numerical notation to express the relationship.

This student's answer to Q5 was: "multiplying the number of white tiles times 2 plus 6 on the sides: x·2+6=x". In other words, S6 used two types of representation: verbal and algebraic, exhibiting a transition from natural to a more general and abstract language.

Note that the three students who generalized spontaneously deduced the general formula by identifying the correspondence relationship in the function f(x)=2x+6.

Most of the 16 students who generalized when prompted (in Q5) expressed the general relationship between the pairs of values (correspondence) verbally. A few representative examples follow.

The students identified the pattern from which they deduced the general formula in a number of ways. In one, eight students described generalization in terms of a rule that in algebraic notation would be represented as f(x)=2x+6. S14, for instance, answered "you get the answer by multiplying the white tiles times 2 and then adding 6". In this case, as in the other seven, generalization was expressed verbally. Student S3, in turn, replied "multiplying the white tiles by two and adding three at the beginning and three at the end". The pattern detected by this student would be represented in algebraic notation as f(x)=2x+3+3. S24 adopted a third approach, identifying the pattern to be f(x)=2(x+2)+2.

One of these students, S1, used primarily verbal representation, although in conjunction with algebraic symbols. In Q5 the answer was "you need to use 2x white tiles +6"; i.e., verbal representation predominated, although with some elements of algebraic symbolism. The implication would seem to be that this student, who used some algebraic symbols sporadically when answering the previous questions, was en route to attaining a more natural and spontaneous use of algebraic symbolism to represent the relationship between variables.

Lastly, the relationship was incorrectly identified by six students in a way that translated to algebraic notation would yield $f(x)=2x+2$. One representative example of this relationship between variables was provided by S9, whose answer to Q5 was "multiply the top and bottom rows by 2 and add 2". Like the other five students, this pupil established a general, albeit mistaken, relationship between the variables.

CONCLUSION

This research supplements other studies focusing on lower grade students' ability to generalize in the context of classroom algebraic functions (e.g., Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015). Here the emphasis was on generalization as deduced by fifth graders.

The tiles problem affords the opportunity to explore students' functional thinking, as it enables fifth graders to progress beyond recursive sequences. In fact, they generalized on the grounds of correspondence and covariation relationships that involved the values of a set of variables.

The overall finding was the existence of two situations in which students generalize: (a) when answering questions about particular (near or far) instances; and (b) when specifically prompted to generalize. Three students generalized spontaneously, i.e., where the question could be answered without doing so. They consequently used generalization as a strategy to reply to questions involving specific circumstances. All the students who established a general relationship between the variables (spontaneously or when prompted) based their deduction on the correspondence relationship.

The students who generalized spontaneously used algebraic notation and verbal representation to express the general relationship between variables. Representation was primarily verbal in students who generalized only when prompted. In line with Blanton, Brizuela, Gardiner, Sawrey, and Newman-Owens (2015), the present authors venture that using algebraic notation would enable students to visualize generalization in fuller detail. That is consistent with the fact that the students who used notation in addition to verbal representation to express relationships did so in questions where generalization was not necessary (spontaneous generalization).

Moreover, the different ways in which students detected patterns in a problem involving a linear function ($f(x)=2x+6$; $f(x)=2x+3+3$; $f(x)=2(x+2)+2$; $f(x)=2x+2$) afforded the opportunity to interpret and understand their thought process when identifying a general relationship between variables.

Lastly, the present findings are related to earlier research results on Spanish fifth graders' ability to generalize (Merino et al, 2013), in which verbal representation was also observed to prevail. This paper describes the general functional relationships detected by students and the questions in which they were identified by functional thinking. These findings support the application of this approach to mathematics teaching in the lower grades, for its favors and enhances algebraic thinking (Blanton, 2008).

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