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Mathematics teachers' conceptual knowledge about and for teaching quadrilaterals

Marcelo Carlos de Proença
Universidade Estadual de Maringá
Maringá — PR, Brasil
mcproenca@uem.br
0000-0002-6496-4912

2238-0345

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Abstract: The aim of the article is to analyze the conceptual knowledge of a group of mathematics teachers about quadrilaterals and the way they practice their teaching. We conducted exploratory research with 23 teachers who answered an online questionnaire, whose results we analyzed descriptively and categorically. The best-known characteristic of quadrilaterals was that it is a plane figure and the least known was that it is a simple figure. Notable quadrilaterals were mentioned more than irregular shapes. In addition, the presence of irrelevant attributes such as thick line and being rotated made it difficult to recognize some figures. In teaching quadrilaterals (n = 17), four teachers would act as expositors of their ideas. Two teachers would not address the non-examples. Eleven teachers would deal with examples and non-examples. In conclusion, training is needed to understand other examples, non-examples and irrelevant attributes to teach in a way that promotes conceptual development.

Keywords: Mathematics Teaching. Geometry. Concept. Exploratory Research.

Conocimiento conceptual de un grupo de profesores de Matemáticas sobre y para la enseñanza de los cuadriláteros

Resumen: El objetivo del artículo es analizar El conocimiento conceptual de um grupo de profesores de matemáticas sobre los cuadriláteros y la forma en que practicansuenseñanza. Realizamos una investigación exploratória con 23 profesores que respondieron a um cuestionario en línea, cuyos resultados analizamos descriptiva y categóricamente. La característica más conocida de los cuadriláteros era ser una figura plana y la menos conocida, ser una figura simple. Se mencionan más los cuadriláteros que las formas irregulares. Además, la presencia de atributos irrelevantes, como la línea gruesa y estar girado, dificulto el reconocimiento de algunas figuras. Em La enseñanza de los cuadriláteros (n =17), cuatro profesores actuaban como expositores de sus ideas. Dos profesores no se acercaron a los no-ejemplos. Once profesores tratarían con ejemplos y no ejemplos. Em conclusión, se necesita formación para comprender otros ejemplos, no ejemplos y atributos irrelevantes conelfin de enseñar de forma que se promueva el desarrollo conceptual.

Palabras clave: Enseñanza de las Matemáticas. Geometría. Concepto. Investigación Exploratoria.

Conhecimento conceitual de um grupo de professores de Matemática sobre e para o ensino de quadriláteros

Resumo: O objetivo deste artigo é analisar o conhecimento conceitual de um grupo de professores de Matemática sobre quadriláteros e a forma de exercerem seu ensino. Realizamos uma pesquisa exploratória com 23 professores que responderam a um questionário *on-line*, cujos resultados foram analisados de forma descritiva e categorial. A característica mais conhecida de quadriláteros foi ser uma figura plana, e a menos conhecida, uma figura simples. Os quadriláteros notáveis foram mais mencionados do que as formas irregulares. Além disso, a presença de atributos irrelevantes, como linha espessa e estar rotacionado, dificultou o reconhecimento de algumas figuras. No ensino de quadriláteros (n =17), quatro professores atuariam como expositores de suas ideias. Dois professores não abordariam os não exemplos. Já 11 professores tratariam de exemplos e não exemplos. Como conclusão, é necessária uma formação para a compreensão de outros exemplos, não exemplos e atributos irrelevantes, de modo a exercerem um ensino que propicie o desenvolvimento conceitual.

Palavras-chave: Ensino de Matemática. Geometria. Conceito. Pesquisa Exploratória.

1 Introduction

The teaching of Geometry in the school must focus, among other aspects, on the learning of conceptual and procedural knowledge, in a manner which the teachers need to be well trained, both about the geometrical concepts to be taught, and the teaching itself (Hoffer, 1983; Schoenfeld, 1986; Clements & Battista, 1992). About the conceptual knowledge, the literature shows the concept is its central study aspect (Pozo, 1998; Klausmeier & Goodwin, 1977; Rittle-Johnson, Schneider & Star, 2015).

The conceptual knowledge in geometry might be understood, in the points of view of (1986) and Ball, Thames and Phelps (2008), as part of the subject knowledge of the Mathematics teacher, in a manner which the quality of this knowledge may reflect, consequently, in the quality of the way of teaching. The study by Steele (2013) investigated the mathematical knowledge of teachers in tasks of area and perimeter, and showed there was more tendency in the usage of procedural knowledge than of conceptual, revealing the need to develop mathematical knowledge for the teaching. Sunzuma and Maharaj (2019) showed that 47,5% (n= 40) of the teachers presented inappropriate knowledge of geometrical content for the teaching, and that 40% (n = 40) demonstrated difficulties in the understanding of geometrical concepts.

In a general manner, difficulties such as those end reaching the learning quality of the students, which reveal difficulties in the usage of conceptual knowledge of Mathematics, permeated by the poor understanding of concepts (Proença & Pirola, 2011; Fernández-Millán & Molina, 2017, 2018; Aydin, 2018; Pereira & Proença, 2019; Gonçalves & Proença, 2020; Proença, Maia-Afonso, Travassos & Castilho, 2020; Scheibling-Sève, Pasquinelli & Sander, 2020).

Facing this fact, we identified the studies relative to conceptual knowledge, specifically of Geometry, tend to focus in higher degree on future teachers, as seen in the investigations by Maia and Proença (2016), Zuya (2017), Castilho and Proença (2018) and Yurniwat and Soleh (2019), Liang and Castillo-Garsow (2020). In the case, researches about teachers, as those of Steele (2013) and Sunzuma and Maharaj (2019), previously cited, indicate the need to widen and deepen the investigation in this subject with active teachers. This is important because it helps to reveal the conceptual knowledge of teachers who act in the classroom. In the present article, assuming as the basis the concept of quadrilaterals, we propose answering the following research question: *What is the conceptual knowledge of a group of Mathematics teachers about quadrilaterals and in which manner they exercise their teaching?*

In order to reach an answer for this question, the objective was analyzing the conceptual knowledge of a group of Mathematics teachers about quadrilaterals and the manner which they exercise their teaching. We structured the article by presenting the theoretical aspects referent to the characteristics of conceptual knowledge and its teaching. Ahead we present the research methodology, the characterization of the group of teachers, the results and the discussions about the conceptual knowledge and its teaching, and, lastly, the conclusion.

2 Conceptual knowledge: characteristics and teaching

The term *conceptual knowledge* is seen knowledge of *concepts*, understood as abstract and general ideas (Rittle-Johnson & Schneider, 2014; Rittle-Johnson, Schneider & Star, 2015). In the view of Klausmeier and Goodwin (1977, p. 312), a *concept* may be described as the "ordained information about the properties of one or more things — objects, events of processes — which makes anything or class of thing to be differentiated from or related to other things or classes of things". For Zabala (1998, p. 27), the *concepts* "refer to the group of facts, objects or symbols which have common characteristics".

This ordained information, as well as these common characteristics pointed by Zabala (1998), it was characterized by Klausmeier and Goodwin (1977) as corresponding to the *defining attributes*, which would be the characteristics which define the concept and, thus, allow generating *examples*. To illustrate this idea, the concept of quadrilaterals can be defined as: figure formed of four sides, sides which are segments of straight line, closed figure and figure which sides only intercept in its vertices (simple figure). Thus, the examples are the squares, rectangles, parallelograms, rhombus, and all the figures which have four sided, because they are derived from the *defining attributes*. In this manner, Klausmeier and Goodwin (1977) also explain an aspect that might be considered about the examples of a concept are their *irrelevant attributes*, in a way which the quadrilateral with their sides represented by thick or thin segments of straight lines, by different sizes, by different colors or rotated constitute characteristics which are not part of the definition of the concept.

As *examples* are part of the concept, of its symbolic representation, Klausmeier and Goodwin (1977) called attention to the need of observing the *perceptibility de examples* and the *numerosity of examples* of a certain concept. For these authors, as in Mathematics, the concepts are abstract, some of them, such as the infinite, are not possible to be perceived by the senses (visualizing, manipulating etc.), but can be represented. Yet to find many examples, the authors pointed this may vary, being that, in the case of the concept of infinite, it would have only one example; and in the case of the concept of real numbers, infinite examples would be presented.

Klausmeier and Goodwin (1977) indicated a concept may be differentiated or related to other concepts, which is corroborated by Hiebert and Lefevre (1986, p. 3), because they also understand the "conceptual knowledge is characterized most clearly as knowledge that is rich in relationships". In this sense, the students may develop a concept from the relations between information which they already have, or even, from new information (Skemp, 1971; Hiebert & Lefevre, 1986). Furthermore, Klausmeier and Goodwin (1977) explained that a learned concept is, many times, in a more elementary level, of the *concrete kind*, in a manner which must be developed along the school year, reaching the *formal* level.

In this same perspective, Hiebert and Lefevre (1986, p. 05) understand that an abstraction occurs in the established relations, in a way which being *primary* is the starting level and that of the *reflexive* kind, a larger degree, "because its construction requires a process of regression and reflection about the information which is being linked. It is in a more elevated level than the primary level, because from its point of view the learner can see much more of the mathematical terrain".

Thus, Klausmeier and Goodwin (1977) pointed that, in order to favor the development of a concept to the students, it is necessary to lead them to make usage of the concept which is being learned, to generalize to other examples of the concept, as well as identifying non-examples. It must also lead them to perceive superordinate and subordinate relations. The first is the relation which parts from the least inclusive to the most inclusive, as in the case of identifying that ‘every square is a rectangle’. On the other hand, the subordinate relation is that which parts from the most inclusive to the least inclusive, as identifying that ‘some rectangle is a square’.

In synthesis, we can say the conceptual knowledge in Mathematics is the knowledge of a concept and its relations to other concepts. Thus, it is different from the procedural knowledge, defined as the knowledge of *procedures* (Rittle-Johnson & Schneider, 2014; Rittle-Johnson, Schneider & Star, 2015), which, in the view of Zabala (1998, p. 43), correspond to “a group of ordained actions with an end, which means, driven to fulfill an objective”. Between the many kinds of procedures, we may cite the driving, cognitive, technical, dexterity, algorithms and the heuristics (Coll & Valls, 1998; Zabala, 1998). Although the learning concepts have relations to the use of knowledge and procedures (Pozo, 1998), the “conceptual knowledge, by our definition, must be learned meaningfully. Procedures, on the other hand, may or may not be learned with meaning” (Hiebert & Lefevre, 1986, p. 08).

Regarding the teaching and learning and the development of concepts, Klausmeier and Goodwin (1977) highlight the teachers need to understand the concepts they teach, in a manner which they know: a) obtaining the definition of concept (mathematical definition); b) identifying the defining attributes; c) identifying some irrelevant attributes; d) identifying examples; e) identifying non-examples; f) identifying the taxonomy (superordinate and subordinate relations); g) identifying some principles (example: every square is formed by segments of straight lines); h) identifying situations for the use of concept; and i) identifying the names of the attributes of the concept.

Thus, the work to be done in the classroom, according to the suggestions by Klausmeier and Goodwin (1977), demands the teacher elaborates a teaching sequence, based on actions such as: i) propose examples and non-examples of concept, so the students search to identify the defining attributes and the irrelevant attributes; ii) ask the students to present a definition (mind construct); iii) present to the students the terminology of the concept and its defining attributes; iv) create articulations between the definition from the students (mind construct) and the mathematical definition (public entity), utilizing the formal symbols; v) asking the students to present new examples, as well as non-examples of the concept; and vi) taking the students to establish superordinate and subordinate relations.

On the contrary of the traditional way of teaching (definition-example-exercises), the work realized in this sequence focuses on the learning and development of concepts, because, according to Klausmeier and Goodwin (1977), it contributes for the students not committing generalization mistakes (overgeneralize or undergeneralize) and of poorly forming the concept. Studies such the one by Proença and Pirola (2011) demonstrate that, sometimes, the students call the cube as square, overgeneralizing the concept of square. Beyond this, this teaching sequence contributes in the sense which values the *mind constructs* of the students, that is, values the learning of concepts from their own individual experiences of learning, so they later articulate the mathematical definition, that is, the concept as *public entity* (Klausmeier & Goodwin, 1977). In this manner, the construction of the concept goes in the direction that “it is not about if the student understands or not, but *how* it is understood” (Coll& Valls, 1998, p. 27, stressing by the authors).

Rittle-Johnson and Schneider (2014) stressed that many studies used diverse kings of tasks to investigate the conceptual knowledge, which could be approached in the classroom.

These tasks may be implicit or explicit. The *implicit* tasks are those used to verify how the students recognize examples of the concept, what could be in the sense of presenting a group of examples and non-examples, indicated by Klausmeier and Goodwin (1977), for the students recognize the examples. These implicit tasks would also help verifying how these students evaluate answers given by other students about what would be the concept. The *explicit* tasks would demand the students know how to generate or select definitions of the concept, or even, that they know how to draw maps which involve characteristics of the concept.

This work to be done in the classroom which involves taking the students to the learning of concepts and to the development of conceptual knowledge is also in the direction of the development of advanced mathematical thinking, pointed by Dreyfus (1991), justly in the strand of concept abstraction. This author indicated that the abstraction of a mathematical concept would be given by the process of representation, of generalization and of synthesis of ideas. The author suggests the usage of more than one representation (in the case, only examples), in a manner which the students may be involved in the generalization, that is, "to derive or induce particulars, to identify commonalities, to expand domains of validity" (Dreyfus, 1991, p. 35), so, in the process of synthesis, make the integration of ideas, pointing a/their definition.

These suggestions of teaching contribute to this because the students are involved in the process of generalization. Therefore, the definitions from these students correspond to their synthesis from the characteristics (defining and irrelevant attributes) which identify in the many representations of the concept (examples and non-examples). As a result, in the view of Hiebert and Lefevre (1986), developing the conceptual knowledge of the students implies in taking them to establish rich relations between the information of a concept, which may reach connections with many other concepts. It is the case, for example, of the students learning the concept of square and its relations with other quadrilaterals, as well as understanding that the square forms the faces of a cube, and not that the cube is a square.

3 Methodology

The study corresponds to an exploratory and descriptive research (Gil, 2012), aiming to obtain and describe the comprehension of knowledge from a group of teachers about the concept of quadrilaterals. As we aimed to obtain this knowledge from teachers from many states of Brazil, we elaborated a questionnaire which was inserted on Google Forms. We sent this form to Mathematics teachers from Brazil, by means of groups in social networks available in March of 2022. After sending then, passing 30 days, we sent again. We obtained, as participants, a group of 23 teachers who act in the teaching of Mathematics.

The Google Form was organized in five parts. In the first, there was the invitation to take part in the research, with the Termo de Consentimento Livre e Esclarecido (TCLE), so they would permit it, in the case they wanted to contribute. The second part asked aspects of the professional profile, to be known: a) the Brazilian State where they act and the graduation; b) time of service as a tenured teacher in the Permanent Board and the level of active teaching; c) the school years of the Final Years of Elementary School and High school which lectured about quadrilaterals. The third and fourth parts contained questions/items which talked about the conceptual knowledge of quadrilaterals by the teachers. The fifth part contained items about how they exercised their teaching, directing justly for the learning of the concept. Board 1, ahead, demonstrates how this instrument of data collection was framed.

Board 1: Questionnaire about conceptual knowledge of quadrilateral and its teaching

About conceptual knowledge:

1) Consider the items below. Mark those which you believe that mast be considered as a form of defining/characterizing what would be a quadrilateral.

( ) It may be a spatial figure.
( ) It is a plane figure.
( ) It is a figure formed by segments of straight lines.
( ) It might be a figure formed by segments of straight lines.
( ) It is a four-sized figure.
( ) It might be a figure with more than four sizes.
( ) It might be an open figure.
( ) It is a closed figure.
( ) It is a simple figure.
( ) There might have intersection between its sides, besides the vertex intersections.
( ) It is a figure which sum of internal angles is 360º.
( ) It might be a figure, which sum of internal angles is different than 360º.

2) Cite all quadrilaterals you know.

3) Mark the options which ARE NOT quadrilaterals.

Figure 1	Figure 2	Figure 3	Figure 4	Figure 5

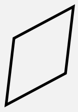
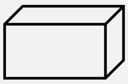
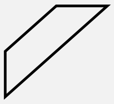
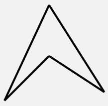
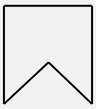
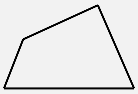
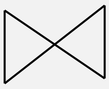
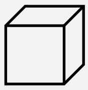
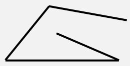
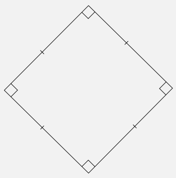
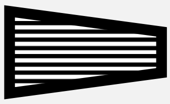
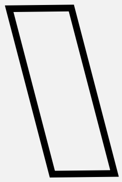
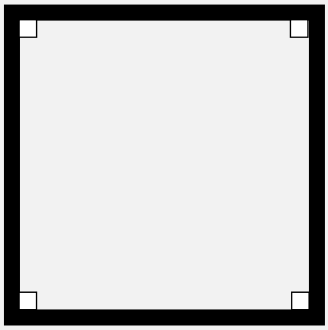
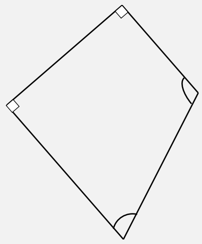
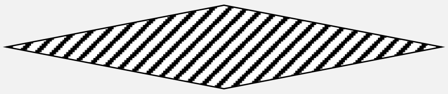
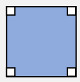
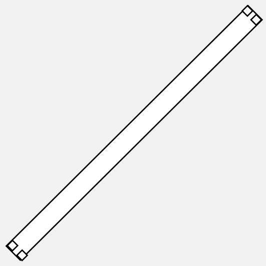

				

Figure 6	**Figure 7**	**Figure 8**	**Figure 9**	**Figure 10**
				

( ) Figure 1 ( ) Figure 2 ( ) Figure 3 ( ) Figure 4 ( ) Figure 5
( ) Figure 6 ( ) Figure 7 ( ) Figure 8 ( ) Figure 9 ( ) Figure 10
( ) All the figures are quadrilaterals

4) Below, we have affirmations about the respective figures. Mark the options you consider correct.

( ) It is a rectangle 	( ) It is about a parallelogram. 	( ) It is a square. 	( ) It is a trapeze. 
( ) It is a parallelogram.	( ) It is a square.	( ) It is a trapeze.	

		
( ) It is a rhombus. 	( ) It corresponds to a square. 	( ) It corresponds to a rectangle. 

About the teaching of quadrilaterals concept:

1) Consider the items described below. Mark next to it with an “X” the order of the actions you adopt (or would adopt) in the teaching of the concept of quadrilaterals.

Items	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th
I present many figures which are quadrilaterals											
I ask the students to present other figures which are not quadrilaterals											
I approach the formulas to calculate areas											
I ask the students to generate a definition, according to their understandings											
I present the mathematical definition											
I approach the characteristics of the figures											
I ask the students to present other figures which would be quadrilaterals											
I present the name of the concept (quadrilaterals)											
I approach the perimeter formula											

I present figures which are not quadrilaterals											
I present both figures which are quadrilaterals, and figures which are not quadrilaterals											

Source: Own authorship

The data were presented in a descriptive and categorical manner (Gil, 2012; Bogdan & Biklen, 1994). These data about the profile of the participants were presented in Charts 1, 2 and 3, in a manner to highlight the characterization of the group of teachers. For the data which involved conceptual knowledge about and the teaching of quadrilaterals, we did the description of the answers in Charts 4 to 8, utilizing the theoretical terms such categories *a priori*. In the case of the aspects of teaching, in Charts 9 and 10, we used categories *a posteriori*, aiming for the understanding of the choices done in the considered orders of teaching.

4 Characterization of the group of teachers

Firstly, we presented a characterization of the Professional profile of the group of 23 teachers. Chart 1demonstrates the Brazilian State and the respective formation in graduation, which contemplated licensed in Mathematics (LM) and licensed in Sciences with Qualification in Mathematics (CHM).

Chart 1: State where they work and formation of graduation

Estado	**LM**	**CHM**	**Total**	**%**
Distrito Federal-DF	1	-	1	4.35
Mato Grosso-MT	1	-	1	4.35
Pará-PA	2	-	2	8.70
Paraíba-PB	-	1	1	4.35
Paraná-PR	9	2	11	47.83
Rio Grande do Sul-RS	2	-	2	8.70
São Paulo-SP	5	-	5	21.74
Total	**20**	**3**	**23**	**100,00**

Source: Own authorship

We observed most of the participants (47.83%) are from the State of Paraná, followed by the State of São Paulo (21.74%). About the formation of graduation, most of them, 20 (86.96%), are licensed in Mathematics. Referent to the time of service as a tenured teacher from the Permanent Board and level of actuation in teaching, Chart 2 shows this characterization.

Chart 2: Time as a tenured teacher and teaching level in which acts

Time (years)	**EF-AF**	**EM**	**EF-AF + EM**	**Total**	**%**
1 to 5	-	2	4	6	26.09
6 to10	3	2	1	6	26.09
11 to 15	2	3	1	6	26.09
16 to 20	1	1	-	2	8.70
21 to 25	1	-	1	2	8.70
26 or more	-	1	-	1	4.35
Total	**7**	**9**	**7**	**23**	**100.00**

Source: Own authorship

We verified that most of the participants had, at the moment, between 1 and 15 years of actuation in school education, totalizing 18 (78.26%) teachers. About the level of actuation, there was a balanced distribution, being larger for those who are only in High school (9; 39.13%). In face of this professional characterization, we also questioned about the school years of Elementary School – Final Years (EF-AF) and High school (EM) which had taught the subject of quadrilaterals in the school. Chat 3 shows the frequency of the items chosen by the participants.

Chart 3: School years in which the participants taught quadrilaterals

School year	Frequency	%
6^{th} grade	15	65.22
7^{th} grade	9	39.13
8^{th} grade	13	56.52
9^{th} grade	11	47.83
EF-AF or EM (*)	8	34.78

(*) Quadrilaterals were reviewed in order to talk about geometrical solids.
Source: Own Authorship

In Chart 3, we observed that most of the teaching approach occurred in the 6th grades of Elementary School (65.22%), followed by the 8^{th} grades of Elementary School (56.52%). An item from the questionnaire, to describe this classroom work, is what pointed about only reviewing quadrilaterals to approach geometric solids in both the considered levels of education, what shows a frequency of answers of 8 (34.78%).

5 Results and discussion

Teacher's conceptual knowledge about quadrilaterals: In this first part, we aim to highlight the participants answer to reveal the conceptual knowledge. Chart 4 shows the items chosen by the teachers which would define/characterize a quadrilateral.

Chart 3: Affirmations chosen to define or characterize what would be a quadrilateral

Nature	Affirmations	Quantity	%
Defining attributes	It is a plane figure	23	100.00
	It is a figure formed of segments of straight lines	21	91.30
	It is a four-sided figure	21	91.30
	It is a closed figure	19	82.61
	It is a simple figure	8	34.78
	It is a figure which sum of internal angles is 360º	21	91.30
Attributes of other concepts	It might be a spatial figure	1	4.35
	It might be a figure formed by segments of straight lines	2	8.70
	It might be a figure with more than four sides	-	-
	It might be an open figure	-	-
	There might be intersection between its sides, besides the vertex intersection.	2	8.70
	It might be a figure which sum of internal angles is different from 360º	1	4.35

Source: Own Authorship

From Chart 4, we observed that, from the affirmations which natures are *defining* *attributes* of quadrilaterals, the 23 teachers recognized "it is a plane figure" as such. We also verified there are teachers which difficulty in the recognition of the totality of *defining attributes* of a quadrilateral, what, in teaching, may lead the students to not identifying the totality of characteristics from quadrilaterals, as Pereira and Proença (2019) showed.

In the view of Hiebert and Lefevre (1986), this teachers' difficulty might be understood as in not reaching the *reflexive* level to abstract relations in this terrain of mathematical quadrilaterals. From the affirmations which are *defining attributes*, this level still needs to be reached about being a simple figure, because it was identified only by 8 (34.78%) teachers. This difficulty may reflect in the students because the study by Proença and Pirola (2009) demonstrated that a low average (M = 42.3%; n = 253) of students from High School identified in an affirmation the feature of simple figure is from the concept of polygon.

About the affirmations which nature involves attributes from other concepts (Chart 4), we observed that 6 participants chose four of them. These choices reveal an understanding which overgeneralizes (Klasumeier & Goodwin, 1977) the concept of quadrilaterals as, for example, indicating that "it might be a spatial figure". This ends by revealing some understanding in the level of *primary* abstraction (Hiebert & Lefevre, 1986), justly for not differentiating the relations between a quadrilateral and other mathematical concepts. If the teacher does not know how to realize this differentiation, it might, consequently, lead the students to not having conditions of perceiving, for example, that a polygon which forms a face of a polyhedron and, thus, polygon is not a polyhedron (Proença & Pirola, 2011).

Chart 5 shows the results referring to the quantities of quadrilateral *examples* which the participants mentioned knowing.

Chart 4: Known quadrilaterals

Quadrilaterals	**Quantity**	**%**
Square	21	91.30
Rectangle	22	95.65
Parallelogram	20	86.96
Rhombus	21	91.30
Trapeze	22	95.65
Irregular	6	26.09

Source: Own Authorship

We observed that the five first examples from Chart 5, provided by the teachers, are notable quadrilaterals, in a manner which there is proximity in the quantity of mentions to these *examples*, maybe for being more known. In front of this fact, the most mentioned quadrilaterals were the rectangle and the trapeze, with 95.65% each. But, we verified none of the five *examples* obtained mention from the total of 23 participants. An explanation may be if they considered the relation of inclusion from the *subordinated* kind (Klausmeier & Goodwin, 1977), as, for example, by citing the trapeze, which parallelogram would be already included.

Chart 5 shows that only 26.09% of the participants mentioned the irregular quadrilaterals, revealing they are able to realize generalization to other examples (Klausmeier & Goodwin, 1977). This result shows the other teachers might have difficulties to exercise a teaching which promotes learning by the students, generating misunderstandings not only to identifying and present *examples* of quadrilaterals (Pereira & Proenca, 2019), but yet for other geometrical concepts, such as, for example, identifying and indicating different kinds of angles (Aydin, 2018).

Chart 6 shows the figures which were considered by the participants as not being examples of quadrilaterals.

Chart 5: Figures considered as not being quadrilaterals

Nature	Items	Quantity	%
Examples of quadrilaterals	Figure 1	1	4.35
	Figure 3	2	8.70
	Figure 5	5	21.74
	Figure 7	1	4.35
Non-examples of quadrilaterals	Figure 2	21	91.30
	Figure 4	19	82.61
	Figure 6	18	78.26
	Figure 8	20	86.96
	Figure 9	21	91.30
	Figure 10	20	86.96

Source: Own Authorship

About the nature of being examples of quadrilaterals (Figures 1, 3, 5 e 7), Chart 6 highlights that nine participants considered wrongly at least once these figures as not being examples of quadrilaterals. The largest quantity of this mistake was for Figure 5 (concave quadrilateral), in a total of 21.74%. This mistake might be due to understanding that an example would be only a notable quadrilateral, with convex shape. In this case, the concave shape may have made it difficult to identify the *defining attributes:* four sides and being a closed figure, in a way to recognize this is an *example* of quadrilateral. In general, it is concerning that there is also the recognition of the Figures 1, 3 and 7 as not being a quadrilateral, because they are examples with, evidently four sides and, thus, should have been generalized as quadrilaterals (Klausmeier & Goodwin, 1977).

For the nature of being non-examples of quadrilaterals, we observe Figure 2 (parallelepiped) and Figure 9 (cube) were the most recognized, with 91.30% each. On the other hand, Figure 6 (concave pentagon) was the least identified as such (78.26%). Possibly, for the other five teachers who did not recognize it as a *non-example* might be due to it being a concave figure. This justification might be the same for those who consider Figure 5 (concave quadrilateral) as *non-example*.

Chart 7 shows the quantity of answers from the participants who considered correct each one of the affirmations about the figures of some quadrilaterals. Such affirmations were grouped, according to the respective *example* of quadrilateral, in order to help to understand this quantity of answers, having in sight the presence of *irrelevant attributes*.

Chart 6: Affirmations about figures considered correct by the participants

Affirmations (all correct)	Irrelevant attributes	Quantity	%
It is a square	Thin and rotated line	23	100.00
It is a square	Thick and large line	20	86.96
Corresponds to a square	Blue inner color and small	19	82.61
It is a rectangle	Green inner color	20	86.96
Corresponds to a rectangle	Thin line and inclined	22	95.65

It is a trapeze	Thick line, hatched and rotated	17	73.91
It is a trapeze	Thin line and rotated	23	100.00
It is a parallelogram	Thin line	20	86.96
It is a parallelogram	Thick line and rotated	17	73.91
It is a rhombus	Hatched	16	69.57

Source: Own Authorship

We observed the 23 teachers recognized the affirmations about the square and the trapeze as corrects, what, according to Klausmerier and Goodwin (1977), shows the present *irrelevant attributes* (sides constituted of thin lines and rotated figures) do not interfere, apparently. In the study by Proença and Pirola (2011), the square with thick line and rotated was recognized by 75.1% (n = 253) of High School students, which shows that, perhaps, in the teaching it is necessary a wider approach about rotated figures and with other irrelevant attributes to potentialize the conceptual learning.

On the other hand, Chart 7 highlights that the smallest percentage of recognition was of 69.57% for the affirmation about being a rhombus, which figure was hatched. Referring to High School students, the study by Proença and Pirola (2011) showed that, when it is about identifying hatched figures, they presented an average below 60%, as in the case of a hatched square which had an average of recognition of 59.7% (n = 253).

In a general manner, the presence of other *irrelevant attributes* indicates difficulty of the participants to recognize the truth in the affirmations. If we observe the other two affirmations about square, we verify that the figure with thick line and larger size (86.96%) and that with blue inner color and of small size (82.61%) were not recognized as correct by everybody. Curiously, the parallelogram with thin line (86.96%), usually treated in the presented position, also was not recognized as correct by everyone.

Teachers' knowledge for the teaching of the quadrilateral concept: In this second part, we search to reveal the form as how the participants conducted the teaching of quadrilaterals. Chart 8 shows the items to teaching, available for the teachers, and the respective quantity for the order they would consider in teaching. We separated these items in two parts, being the six first in the order which reflects to that indicated by Klausmeier and Goodwin (1977), for a work which leads the students to the conceptual formation.

Chart 8: Order presented to the teaching to be realized

Items	**Order chosen by the participants (Qtd) (n = 23)**										
	1st	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**	**8th**	**9th**	**10th**	**11th**
1st) I present both figures which are quadrilaterals and figures which are not quadrilaterals	8	3	1	2	0	0	0	0	1	1	1
2nd) I ask the students to generate a definition, according to their understandings	1	3	3	2	3	3	1	2	0	0	0
3rd) I present the name of the concept (quadrilaterals)	1	1	1	2	2	2	1	3	0	0	1
4th) I present the mathematical definition	0	1	2	2	2	0	3	1	3	1	1
5th) I ask the students to present other figures which would be quadrilaterals	0	1	2	1	4	1	2	0	1	0	0

6th) I ask the students to present other figures which would not be quadrilaterals	0	4	0	1	2	3	1	0	0	1	0
I present many figures which are not quadrilaterals	9	1	2	3	1	0	0	0	0	0	1
I approach formulas to calculate areas	0	0	1	0	0	1	0	1	4	2	2
I approach the characteristics of the figures	1	6	5	0	1	3	0	0	0	0	1
I approach the perimeter formula	0	0	1	0	0	0	1	2	1	3	2
I present the figures which are not quadrilaterals	2	1	2	1	0	1	1	3	1	1	0

Source: Own Authorship

From Chart 8, we observed that there is a trend in utilizing the item which involves both *examples* and *non-examples* (Klausmeier & Goodwin, 1977) in the beginning of the teaching, in a total of 8 teachers (34.78%), as well as utilizing the item which involves only the *examples*, in a total of 9 (39.13%). The choice for one or other path is adequate, because, according to Klausmeier and Goodwim (1977), it makes it possible to involve the students in the identification of *relevant* and *irrelevant attributes*. In the view of Dreyfus (1991), treating only the representation of the concept (only examples) helps in the process of abstraction.

In front of it, we aim to point the categorization of a teaching sequence which resulted from these two largest choices (n =17). Chart 9 demonstrates the teaching sequence which derives from the initial order being: “I present both figures which are quadrilaterals and figures which are not quadrilaterals”.

Chart 7: Teaching sequence based on the initial usage of examples and non-examples

Teaching	Participants	Quantity (n = 8)	%
Only the techer explains/defines	P7 and P8	2	25.00
Values the work with examples and non-examples in order to late ask the students to generate a definition	P13, P15, P16, P18, P19 and P22	6	75.00

Source: Own Authorship

We verified two teachers (P7 and P8), although indicating bringing *examples* and *non-examples* of quadrilaterals, the teaching order focused only in the teaching explaining/exposing the conceptual aspect. In order to illustrate, it follows the order presented by P8: 2nd — I present the name of the concept (quadrilaterals); 3rd — I approach the characteristics of the figures; 4th — I present the mathematical definition; 5th — I ask the students to generate a definition, according to their understanding. In the view of Rittle-Johnson and Schneider (2014), this teaching order would be an *explicit* task which aims to give the students conditions of generating a definition, but not constructive, because it occurs only based on what the teacher exposes.

Therefore, this order ends by not allowing the students to generate a definition as a *mind construct* (Klausmeier & Goodwin, 1977), because there is little possibility of involving them in the acknowledgement of *defining attributes* and noticing *irrelevant attributes* from their previous knowledge. This may have relation to a structural teaching difficulty about the mathematical knowledge for the teaching of geometry, according to what was revealed in the studies by Steele (2013) and Sunzuma and Maharaj (2019).

Contrary to this, Chart 9 shows six teachers (P13, P15, P16, P18, P19 and P22) valued a teaching order which valued giving voice to the students, before asking them to generate a definition (mind construct). This order occurred in the following manners:

- P16 would ask then the item of the order indicated by Klausmeier and Goodwin (1977): 2nd — I ask the students to generate a definition, according to their understanding; 3rd — I approach the characteristics of the figures.
- Different from P16, the participant P15 would ask, before, for the students to mention more *examples* and *non-examples*: 2nd — I ask the students to present other figures which are not quadrilaterals; 3rd — I ask the students to present other figures which would be quadrilaterals; 4th — I ask the students to generate a definition, according to their understanding; 5th — I present the mathematical definition.
- P13, P18, P19 and P22 tended to approach the characteristics and then ask the students to present *examples* and *non-examples*, as well as the teachers themselves presenting examples. The order by P19 illustrates this result: 2nd — I approach the characteristics of the figures; 3rd — I present many figures which are quadrilaterals; 4th — I ask the students to present other figures which would be quadrilaterals; 5th — I ask the students to present other figures which are not quadrilaterals; 6th — I ask the students to generate a definition, according to their understanding.

In what refers to the second trend of beginning of the teaching order (Chart 8), Chart 10 shows the teaching sequences which derive from the initial order being: “I present many figures which are quadrilaterals”.

Chart 8: Teaching sequence based on the initial usage only of examples

Teaching	Participants	Quantity (n = 9)	%
Only the teacher explains/defines	P4 and P5	2	22.22
Asks the students to generate a definition (based on the examples) and does not approach non-examples	P9 and P17	2	22.22
Includes work with non-examples and later asks the students to generate a definition	P2, P3, P6, P11 and P12	5	55.56

Source: Own Authorship

After bringing only *examples* of quadrilaterals, P4 and P5 would act being content expositors, according P5 indicated by pointing only the following items: 2nd — I present the mathematical definition; 3rd — I present figures which are not quadrilaterals. In the perspective of Rittle-Johnson and Schneider (2014), this posture reveals the lack of attitude to propose an *explicit* task of generating a definition by part of the students. According to Klausmeir and Goodwin (1977), there would be little opportunity for the students constitute their definitions as *mind construct*. In this sense, the teaching posture by P5 tends to reach other subjects, in a manner which the students may even propose examples, but they might not perceive the present mathematical structure, due to the inconsistence of the worked conceptual knowledge (Fernández-Millán & Molina, 2018; Scheibling-Sève, Pasquinelli & Sander, 2020).

On the other hand, P9 and P17 advance a little, by indicating asking the students to generate a definition, but do not treat about *non-examples*, according only to those two items ahead which P9 indicated: 2nd — I ask the students to generate a definition according to their understanding; 3rd — I ask the students to present other figures which would be quadrilaterals. We understand it is important to approach *non-examples*, in order to potentialize the students understanding of the concept of quadrilaterals at *formal* level (Klausmeier & Goodwin, 1977), justly derived from the abstraction of the concept by the relations which may establish the *reflexive* (Hiebert & Lefevre, 1986).

Contrary to these postures, P2, P3, P6, P11 e P12 also involved a teaching which valued the usage of *non-examples*, before asking the students to generate a definition (mind construct). The order by P12 illustrates this result, which corresponds to five items: 2nd — I ask the students to present other figures which are not quadrilaterals; 3rd — I approach the characteristics of the figures; 4th — I present both figures which are quadrilaterals and figures which are not quadrilaterals; 5th — I ask the students to generate a definition according to their understanding.

6 Conclusion

The present article has as objective analyzing the conceptual knowledge of a group of Mathematics teachers about quadrilaterals and the manner they exercise their teaching. By means of a *Google Form*, 23 teachers from Brazil answered the questionnaire. In what regards the conceptual knowledge of quadrilaterals, we consider that, in terms of *defining attributes*, there was a recognition of attributes in higher degree, except that of being a simple figure, what still needs to be understood. The characteristics of being a spatial figure and having the sum of all its inner angles different than 360º as not being defining feats of quadrilaterals.

In terms of knowing *examples* and *non-examples*, there was a higher incidence in mentioning notable quadrilaterals and few about those of irregular form, and, even, mentioning those of concave shape. In this sense, our data show recognizing a concave figure as being or not a quadrilateral indicated that possibly the concave shape generated doubts. Besides this, figures as the rhombus and the trapezium were mistakenly recognized by five teachers as a *non-example*, possibly because they were inclined, which is an *irrelevant attribute*.

About recognizing the *examples* having in sight the presence of *irrelevant attributes* in the presented figures, we found the presence of a thick line, of being hatched or being rotated indicated more difficulty by the teachers in recognizing each one of the figures. In the case of the square, one of the *examples* of most known quadrilaterals, it shows this difficulty in some teachers in stick to the *defining attributes*, and not to the *irrelevant attributes*.

About the teaching of the quadrilateral concept, we found the tendency was of exercising the beginning of this teaching, both by the usage of *examples* and *non-examples*, and by the usage only of the *examples* (n = 17). Independently of adopting one of those initial paths, four teachers revealed following a traditional way of teaching, regarding for the posture of the teacher as exhibitor of ideas. We also found that two teachers, besides raising a debate about the characteristics of the *examples*, did not mention approaching the *non-examples*. In total, 11 teachers revealed treating both examples, and non-examples in the order of their teaching sequence.

We concluded that there was recognition of aspects which involved the concept of quadrilaterals by part of most of the teachers who took part in the research. Nevertheless, there is the need of searching to understand more about *examples* and *non-examples*, and the attention to the fact that the presence of *irrelevant attributes* do not interfere in the concept. In the case of exercising the teaching, there are teachers with different postures, being noted the traditional forma, with basis only in the role of expositor of the teacher. An important aspect is the need of those teachers to incorporate, in their teaching, the work with the *non-examples*, once it may potentialize the conceptual development.

About the limitations of our study, we had the intention of obtaining a bigger number of participant teachers, because we relied on the social networks and, in this way, they would be able to choose when to answer the online questionnaire. If we had more teachers, it would be possible presenting these results with a more robust panorama about the conceptual aspect of quadrilaterals and its teaching. Besides the time we sent repeatedly the questionnaires having been enough, we suggest that, for future studies, it would be important for the period to be even longer. On the other hand, it may be that, even widen this duration, it does not occur a considerable increase, having in sight many teachers are not interested in answering online questionnaires of do not feel comfortable in answering about Geometry, due to some difficulties about the subject, without intention of exposing them.

In a general manner, our study contributes in the field of research by widening the results about the aspects of quadrilaterals concept in the scope of conceptual knowledge facing the few studies of the subject. In the case, treating the *non-examples* and the *irrelevant attributes* revealed with more deepness how the teachers handled the recognition of *examples* of quadrilaterals. Thus, studies can be done, by mean of teacher training proposals which incorporate the usage of *non-examples* and the focus on *irrelevant attributes*. It is possible, then, to analyze the construction of conceptual knowledge, as well as analyzing how these teachers would forward teaching proposals to the formation and the conceptual development of the students.

References

Aydin, U. (2018). Conceptual and procedural angle knowledge: do gender and grade level make a difference? *International Journal for Mathematics Teaching and Learning*, *19*(1), 22-46.

Ball, D. L.; Thames, M. H. & Phelps, G. (2008). Content Knowledge for Teaching. *Journal of Teacher Education*, *59*(5), 389-407.

Bogdan, R.; Biklen, S. (1994). *Investigação Qualitativa em Educação*. Uma introdução à teoria e aos métodos. Translation by M. J. Alvarez; S. B. Santos & T. M. Baptista. Porto: Porto Editora.

Castilho, G. R.& Proença, M. C. (2018). Análise do conhecimento de licenciandos em matemática sobre o conceito e o ensino de polígono. *Revista Prática Docente*, *3*(1), 32-49.

Clements, D. H.& Battista, M. T. (1992). Geometry and spatial reasoning. In: D. A. Grouws(Ed.). *Handbook of research on mathematics teaching and learning*: *a project of the National Council of Teachers of Mathematics*. (pp. 420-464). New York: Macmillan.

Coll, C.& Valls, E. (1998). A aprendizagem e o ensino dos procedimentos. In: C. Coll; J. I. Pozo; B. Sarabia & E. Valls (Org.). *Os conteúdos na reforma: ensino e aprendizagem de conceitos, procedimentos e atitudes*. Translation by B. A. Neves (pp. 73-118). Porto Alegre: Artes Médicas.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In: D. Tall (Ed.). *Advanced mathematical thinking*. (pp. 25-41). Dordrecht: Kluwer.

Fernández-Millán, E. & Molina, M. (2017). Secondary students' implicit conceptual knowledge of algebraic symbolism: an exploratory study through problem posing. *International Electronic Journal of Mathematics Education*, *12*(3), 799-826.

Fernández-Millán, E. & Molina, M. (2018). Ejemplos y definiciones de ecuaciones: una ventana haciaelconocimiento conceptual de estudiantes de secundaria. *PNA*, *12*(3), 147-172.

Gil. A. C. (2012). *Métodos e técnicas de pesquisa social*. 6. ed. São Paulo: Atlas.

Gonçalves, B. M.&Proença, M. C. (2020). Análise dos conhecimentos conceitual e procedimental de alunos do primeiro ano do ensino médio sobre equação do 2.º grau. *Revista Sergipana de Matemática e EducaçãoMatemática*, *5*(2), 209-228.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: an introductory analysis. In: J. Hiebert (Ed.). *Conceptual and procedural knowledge: the case of mathematics.* (pp. 1-27). Hillsdale, NJ: Erlbaum.

Hoffer, A. (1983). Van Hiele-based research. In: R. Lesh & M. Landau (Eds.). *Acquisition of mathematics concepts and processes.* (pp. 205-227). New York: Academic Press.

Klausmeier, H. J. & Goodwin, W. (1977). *Manual de Psicologia Educacional: aprendizagem e capacidades humanas*. Translation by M. C. T. A. Abreu. São Paulo: Harper & Row.

Liang, B. & Castillo-Garsow, C. (2020). Undergraduate Students' meanings for central angle and inscribed angle. *The Mathematics Educator*, *29*(1), 53–84.

Maia, E. J. & Proença, M. C. (2016). A resolução de problemas no ensino da geometria: dificuldades e limites de graduandos de um curso de pedagogia. *Revista Eletrônica de EducaçãoMatemática*, *11*(2), 402-417.

Pereira, M. C. G. & Proença, M. C. (2019). O conceito de quadriláteros: análise doconhecimento de quatro alunos do sétimo ano do ensino fundamental. *Educação Matemática em Revista*, *24*(62), 108-124.

Pozo, J. I. (1998). A aprendizagem e o ensino de fatos e conceitos. In: C. Coll;J. I. Pozo;B. Sarabia & E. Valls (Orgs.). *Os conteúdos na reforma: ensino e aprendizagem de conceitos, procedimentos e atitudes*. Translation by B. A. Neves. (pp. 17-71). Porto Alegre: Artes Médicas.

Proença, M. C.; Maia-Afonso, E. J.; Travassos, W. B. & Castilho, G. R. (2020). Resolução de Problemas de Matemática: análise das dificuldades de alunos do 9.º ano do ensino fundamental. *Amazônia: Revista de Educação em Ciências e Matemáticas*, *16*(36), 224-243.

Proença, M. C. & Pirola, N. A. (2011). O conhecimento de polígonos e poliedros: uma análise do desempenho de alunos do ensino médio em exemplos e não-exemplos. *Ciência & Educação*, *17*(1), 199-217.

Proença, M. C. & Pirola, N. A. (2009). Um estudo sobre o desempenho e as dificuldades apresentadas por alunos do ensino médio na identificação de atributos definidores de polígonos. *Zetetiké*, *17*(31), 11-45.

Rittle-Johnson, B. & Schneider, M. (2014). Developing conceptual and procedural knowledge of mathematics. In: R. C. Kadosh & A. Dowker (Eds.). *Oxford handbook of numerical cognition.* (pp. 1118–1134). Oxford, UK: Oxford University Press.

Rittle-Johnson, B.; Schneider, M. & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review*, *27*(4), 587–597.

Scheibling-Sève, C.; Pasquinelli, E. & Sander, E. (2020). Assessing conceptual knowledge through solving arithmetic word problems. *Educational Studies in Mathematics*, *103*(3), 293-311.

Schoenfeld, A. H. (1986). On having and using geometric knowledge. In: J. Hiebert (Ed.). *Conceptual and procedural knowledge: the case of mathematics.* (pp. 225-264). Hillsdale, NJ: Erlbaum.

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational*

Researcher, *15*(2), 04-14.

Skemp, R. R. (1971). *The psychology of learning mathematics.* Middlesex, England: Penguin.

Steele, M. D. (2013). Exploring the mathematical knowledge for teaching geometry and measurement through the design and use of rich assessment tasks. *Journal of Mathematics Teacher Education*, *16*(4), 245-268.

Sunzuma G. & Maharaj A. (2019). In-service Teachers' geometry content knowledge: implications for how geometry is taught in teacher training institutions. *International Electronic Journal of Mathematics Education*, *14*(3), 633-646.

Yurniwati, Y. & Soleh, D. A. (2019). Geometric conceptual and procedural knowledge of prospective teachers. *International Journal of Education and Pedagogy*, *1*(2), 108-117.

Zabala, A. (1998). *A prática educativa: como ensinar*. Translation by E. F. F. Rosa. Porto alegre: ArtMed.

Zuya, H. E. (2017). Prospective Teachers' conceptual and procedural knowledge in mathematics: the case of algebra. *American Journal of Educational Research*, *5*(3), 310-315.