Professional learning tasks and mathematical knowledge involving the algebraic structure of Groups: an experience in the degree in Mathematics teaching

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Abstract: Considering the teacher education of prospective teachers, the objective of this article is to understand how professional learning tasks are carried out in classes of a course of Algebra in the teacher education. Using Design-Based Research cycles, documents and video recordings were collected throughout the course of the classes. Data were analyzed in order to interpret how future teachers mobilize School-Related Content Knowledge and how teacher educators generate professional learning opportunities for them. The results indicate the possibility of approaching academic mathematics in specific-content disciplines of the teacher education, focusing on teacher education and the changes that future teachers' knowledge may undergo during training. It is concluded that the use of resources that explore connections between academic and school mathematics, as was the case with formative tasks, seems to be a way to promote mathematical and didactic discussions in the teaching degree.

Keywords: Prospective Teacher Education. Academic Mathematics. Mathematics Teaching. Professional Learning. Algebra.

Tareas de aprendizaje profesional y conocimiento matemático involucrando la estructura algebraica de Grupos: una experiencia en la Licenciatura en Matemáticas

Resumen: Teniendo en cuenta la formación inicial de los profesores, este artículo tiene como objetivo comprender cómo se llevan a cabo las tareas de aprendizaje profesional en las clases de una asignatura de Álgebra en la carrera de Matemáticas. Usando ciclos de Investigación Basada en Diseño, la investigación recopiló documentos y grabaciones de video del desarrollo de la clase. Los datos se analizaron para interpretar cómo los futuros maestros movilizan el conocimiento del contenido relacionado con la escuela y cómo los capacitadores generan oportunidades de aprendizaje profesional. Los resultados indican la posibilidad de abordar la matemática académica en disciplinas específicas de la carrera, considerando la formación docente y cómo se pueden modificar los saberes de los futuros docentes durante el proceso de formación. Se concluye que el uso de recursos que exploren conexiones entre las matemáticas académicas y escolares puede ser una forma de promover discusiones matemáticas y didácticas

en la carrera docente.

Palabras clave: Formación Inicial Docente. Matemáticas Académicas. Enseñanza de las Matemáticas. Aprendizaje Profesional. Álgebra.

Tarefas de aprendizagem profissional e o conhecimento matemático envolvendo a estrutura algébrica de Grupos: uma experiência na licenciatura em Matemática

Resumo: Considerando a formação inicial de professores, o objetivo neste artigo é compreender como tarefas de aprendizagem profissional são realizadas em aulas de uma disciplina de Álgebra na licenciatura em Matemática. Utilizando ciclos de *Design-Based Research*, foram coletados documentos e gravações em vídeo ao longo do desenvolvimento das aulas. Os dados foram analisados de modo a interpretar como os futuros professores mobilizam o Conhecimento do Conteúdo Relacionado à Escola e como as formadoras geram oportunidades de aprendizagem profissional para eles. Os resultados indicam a possibilidade de abordar a matemática acadêmica em disciplinas específicas da licenciatura, tendo como foco a formação docente e as modificações que os conhecimentos dos licenciandos podem sofrer durante a formação. Conclui-se que o uso de recursos que exploram conexões entre a matemática acadêmica e escolar, como foi o caso das tarefas formativas, parece ser um caminho para a promoção de discussões matemáticas e didáticas na licenciatura.

Palavras-chave: Formação Inicial de Professores. Matemática Acadêmica. Ensino de Matemática. Aprendizagem Profissional. Álgebra.

1 Introduction

The understanding of mathematical knowledge no longer as unique, led to the use of terms such as "school mathematics and academic mathematics" (Moreira & David, 2008; Klein, 2009). At the same time, more recent research has sought to clarify the connection between these two forms of knowledge and the benefits they can bring to the teaching profession (Wasserman, 2016).

Regarding teachers, Fiorentini and Oliveira (2013) argue that they should know mathematics in a deep and diversified way, going beyond the scientific field and reaching, above all, school mathematics. This is because such knowledge can provide the development of a meaningful mathematics class for students, leading them to reach a connection between what is learned by them and what was historically produced. In this way, mathematical knowledge for teacher education must be directed to teaching practice and oppose the collecting view of formulas and procedures, as studies related to the subject have pointed out, as indicated by Patrono and Ferreira (2021).

In this sense, professional knowledge, which goes beyond understanding the content and must consider the context in which the teacher is inserted. This requires even greater skills on the part of teacher educators (Fiorentini & Oliveira, 2013; Ribeiro & Ponte, 2020; Aguiar, Doná, Jardim & Ribeiro, 2021a), making it necessary to rethink teacher education according to the needs of the profession (Antunes, Cambrainha, Moustapha-Corrêa & Matos, 2021). An example would be to approach the contents of academic mathematics with a focus on school mathematics, aiming to improve teacher education (Wasserman, 2017). However, this can be a challenge when it comes to specific disciplines, such as Algebra (Zazkis & Leikin, 2010).

Therefore, this article seeks *to understand how professional learning tasks are carried out in classes of an Algebra course in the Mathematics degree.* For this, we seek to answer two

questions: i) How are professional knowledge modified by future teachers in carrying out professional learning tasks in an Exploratory Teaching approach? and; ii) What practices and how do teachers educators use them to generate professional learning opportunities when they seek to articulate mathematical knowledge of the algebraic structure of Groups to the teaching of school contents?

Given this, in the next section, we will present the theoretical basis on knowledge and opportunities for professional learning in teacher education, also highlighting the importance of algebra and its connections with teaching. The objective is to provide support for the analysis of a set of lessons given to two Algebra courses, addressing the theme of "Algebraic Structure of Groups".

2 Mathematical Knowledge and prospective teacher education

Continuing the ideas presented by Shulman (1986) about the knowledge needed to teach, new studies have been considered in Brazil in relation to this theme. Some of them are related to specific models of knowledge of teachers who teach mathematics (Patrono & Ferreira, 2021). Among these models, the Mathematical Knowledge for Teaching (MKT) stands out, combining knowledge, teaching practice and student learning, allowing to understand how mathematical knowledge is mobilized in teaching (Ball, Thames & Phelps, 2008).

However, according to Speer, King and Howell (2015), the six MKT domains do not shed light on the relationship between academic and school mathematics, since the studies that originated these domains observed teachers working in contexts corresponding to the initial and final years of Brazilian elementary education. The same authors argue that the domain of Common Content Knowledge (CCK), related to knowledge of mathematics in a more general way, and Specialized Content Knowledge (SCK), exclusive to teaching, seem not to be sufficient to address the mathematical knowledge mobilized by teachers who teach in high school or higher education. Despite the similarities in teaching at different levels, Speer, *et al*. (2015) indicate that the diversity of approaches and the use of definitions with the advancement in schooling, up to higher education, point to a differentiated use of academic mathematical knowledge to support teaching, which requires a better understanding.

To frame a type of knowledge necessary for teachers working in high school, Dreher, Lindmeier, Heinze and Niemand (2018) present School-Related Content Knowledge (SRCK), which considers the non-trivial relationships between School Mathematics (SM) and Academic Mathematics (AM), and is divided into three facets (Figure 1):

i) Knowledge about the curricular structure and its legitimation in the sense of meta-mathematical reasons: deals with the dialectic relationship between academic and school mathematics, approached from fundamental ideas that allow the inclusion of certain contents in the school curriculum.

ii) Knowledge of the relationship between SM and AM: explores how school mathematics topics are grounded in academic mathematics, thus revealing theorems, proofs and ideas that are behind school content.

iii) Knowledge of the relationship between AM and SM: it is based on the transformation of academic-level mathematical content, for didactic purposes, decompressing or trimming academic mathematics into school mathematics.

SRCK shares knowledge with mathematicians and does not mix with pedagogical knowledge with regard to students' mistakes, for example. Thus, it is characterized by a specific look at mathematics, necessary for the development of the teacher's mathematical knowledge

and which can influence decision-making of a mathematical nature and its teaching.

Figure 1: Conception of School-Related Content Knowledge (SRCK)
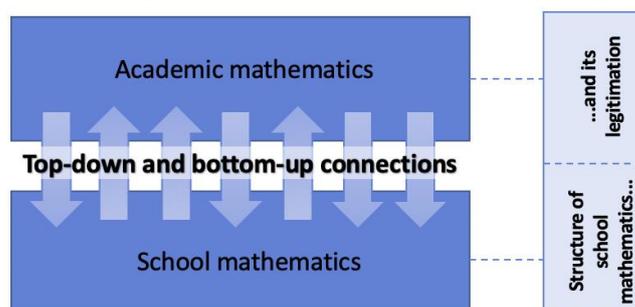
Source: Adapted from Dreher *et al*. (2018, p. 330)

Although the prospective teacher education of Mathematics teachers in Brazil presents differences with the context in which the MKT was originated, it is believed that, by considering the SRCK, it is possible to unveil and understand how mathematical knowledge can be mobilized by future teachers in a degree course in Mathematics, which provides specific preparation to work in the final years of Elementary and High School.

3 Abstract Algebra for teaching

Algebra is an area of mathematics present in Basic Education, "since it is a language widely used in various fields of Mathematics, as well as in other sciences" (Silva, Mondrini, Mocrosky & Pereira, 2021). It can be explored in conjunction with numbers and geometry. Despite the different conceptions of algebra presented by teachers and their educators, it is important to understand the teaching and learning of algebra through clear and objective connections, in order to break with technical models still found in mathematics lessons (Ribeiro, 2016; Silva *et al.*, 2021).

On the other hand, Wasserman (2016) suggests that the teaching of school mathematics, justified by Abstract Algebra, can change teachers' understanding and promote influences in teaching. Zazkis and Marmur (2018) use advanced mathematics knowledge to exemplify elementary mathematical ideas in initial teacher education courses, which can be supported by the algebraic structure of Groups (ASG). In both studies, the treatment of algebraic properties such as associativity, commutativity of operations on different sets and the relationship between an element, its symmetric (or inverse) and neutral in a given set equipped with an operation, as well as the use of the subscript -1 to denote the symmetrical element (Wasserman, 2017; Zazkis & Kontorovich, 2016).

In the study by McCrory, Floden, Ferrini-Mundy, Reckase and Senk (2012), we find Mathematical Knowledge for Teaching Algebra, which in addition to dealing with mathematical knowledge per se, explores practices in the mathematical use of knowledge in teaching, which was also addressed by Gonçalves, Ribeiro and Aguiar (2022). According to McCrory *et al*. (2012), there are three practices that can help in understanding and evaluating teachers' knowledge for teaching algebra: *connecting* topics, representations, and domains of algebra; *trimming* the complexity of an advanced math topic to gain understandings of school math; and *unpacking* hidden meanings in formulas and procedures, as well as directing and clarifying possible constraints.

Although McCrory *et al*. (2012) and Gonçalves *et al*. (2022) have explored practices in basic school contexts, we believe that such practices can be explored in teacher education in which Algebra is discussed. This is how we used the framework in our study.

4 Professional Learning Opportunities and Exploratory Teaching in Teacher Education

Professional learning opportunities (PLO) are characterized by "collective moments in which teachers work and discuss mathematical and didactical situations in order to expand their professional knowledge for teaching" (Ribeiro & Ponte, 2019, p. 50) and use professional learning tasks as a key element to generate such opportunities.

The same authors, in another study (Ribeiro & Ponte, 2020), present a theoretical-methodological model called Professional Learning Opportunities for Teachers (PLOT Model), (Figure 2), which can be used to organize educational processes with teachers, as well as how to evaluate the promotion of learning opportunities from the integration of three domains: Role and Actions of the Teacher Educator (RATE), Professional Learning Tasks for Teachers (PLTT) and Discursive Interactions Among Participants[1] (DIAP). According to Ribeiro and Ponte (2020), each of the domains has two conceptual and two operational components, which will be better explored below.

Figure 2: PLOT Model
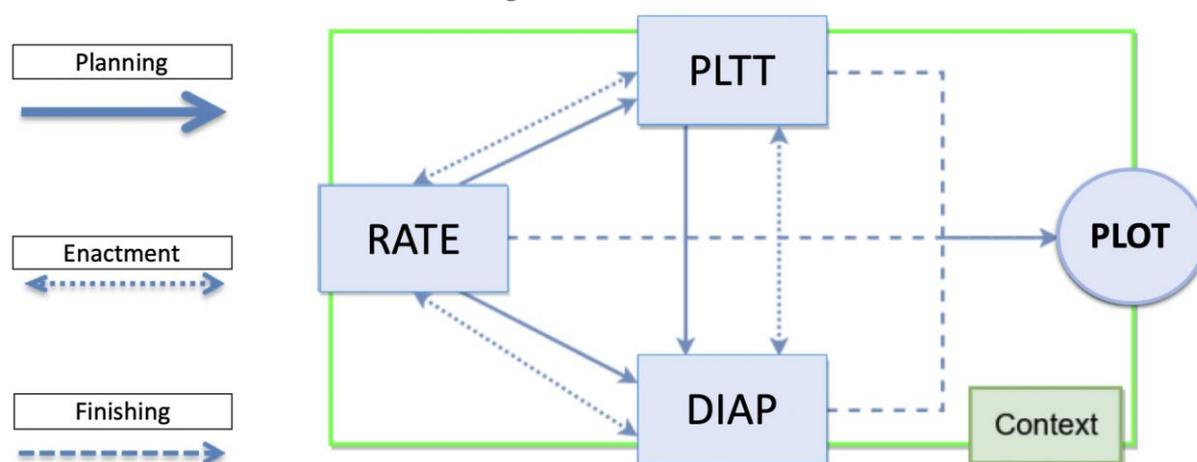
Source: Adapted from Ribeiro and Ponte (2020)

The RATE domain, explored in detail in the work by Aguiar *et al*. (2021a), contemplates in the conceptual dimension the *Approximation* between school mathematics and academic mathematics (and vice versa), taking as reference the ideas pointed out by Moreira and David (2008); and the *Articulation* component, which considers the relationships between the mathematical and didactical dimensions of Professional Knowledge (Ponte, 1999). In the operational dimension, there are the components *Management* of an exploratory teaching environment (Ponte & Quaresma, 2016) and *Orchestration* of mathematical and didactical discussions (Borko, Jacobs, Seago & Mangram, 2014).

In the PLTT domain, the conceptual dimension is composed of *Professional Knowledge* components, characterized by mathematical and didactical knowledge to teach in an integrated way (Silver, Clark, Ghousseini, Charalambous & Sealy, 2007); and *Inquiry-based Approach*, which directs a educational process in order to provide participants with an exploratory teaching and learning environment (Canavarro, Oliveira & Menezes, 2012). In the operational dimension, the PLTT has as a component the *Mathematical Task* at school level and the *Records of Practice*, which are evidence of the application of the mathematical task in basic school and other materials related to teaching (Aguiar, Ponte & Ribeiro, 2021b).

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[1] As it is outside the scope of this article, we will not explore the IDP domain, which is detailed by Trevisan, *et al*., (2023, in press)

In addition to a conceptual structure that supports the design of educational processes, it is necessary to consider the importance of involving teachers and future teachers in the planning of classes to be held in Basic Education, which use Inquiry-based Approach (Aguiar *et al.*, 2021b; Trevisan, Ribeiro & Ponte, 2020; Trevisan, Silva, Silva & Ribeiro, 2023). However, understanding how the development of classes for future teachers occurs is still something to be better explored, since "the environment in which mathematics is learned defines the knowledge that is created" (Ticknor, 2012, p. 309).

Inquiry-based Approach provides a teaching and learning environment with moments of autonomous work organized in stages of introduction, realization, discussion and systematization of a task (Canavarro *et al.*, 2012). In turn, the management of this teaching environment can be carried out with the support of practices that facilitate mathematical discussions of these tasks, such as the five pedagogical practices presented by Stein, Engle, Smith & Hughes (2008): anticipate, monitor, select, sequence, and connect student responses.

In a similar way, Borko *et al.* (2014) present a set of practices to manage video-based discussions in teacher education and divide them into two parts: planning and orchestration. Practices oriented to planning include the definition of objectives, the identification of resources and the elaboration of guiding questions for the discussion, which were discussed in Jardim, Ribeiro e Aguiar (2023). As for the essential practices for orchestrating discussions, according to Borko *et al.* (2014), focus on helping teachers observe resources related to the mathematical content addressed, students' thinking and didactical aspects observable on video. According to the authors, the first practice aims to awaken the teachers' thinking about the topic addressed, which can be done by encouraging them to present a descriptive comment and, thus, help them to detail the actions observed in the video. From this, the second practice proposes that the teacher educator guides and supports the elaboration of ideas, looking for evidence in the allegations presented by the participants. Finally, the third practice aims to help teachers connect them analyzes to the main mathematical and didactical ideas initially aimed at.

Although such practices are proposed to explore videos, it seems to us that they can be used to explore other practical resources, such as protocols with student resolutions when solving tasks, reports or class episodes, among others, since the use of these artifacts presents similar purposes in teacher education, which is to promote teacher learning (Ticknor, 2012; Ribeiro & Ponte, 2019; Aguiar *et al.*, 2021b).

5 Context of the Study

The Mathematics degree course at the Federal Institute of São Paulo, São Paulo campus, offers the discipline entitled "Algebra" for students who are in the 6th semester of the course, and addresses algebraic structures, such as Rings, Fields and Groups. It is in this context that two PLTTs were performed, both addressing definitions and properties related to ASG. The first (PLTT-1) had the potential to make connections between different mathematical contents of basic school (rational numbers and matrices) and the second (PLTT-2) explored the concept of function at different school levels through mathematical tasks.

To develop them, the exploratory teaching approach was used, and the educational process took place in three stages: the introduction using an initial task (IT); carrying out the PLTT in small groups (SG); and, finally, discussion and systematization in plenary sessions. In the first stage, future teachers (FT) individually solved an initial task consisting of five mathematical tasks involving mathematical content of Basic Education. The task was complemented by questions entitled "For the reflection of the Future Teacher", which aimed at the individual reflection of the FT on the mathematical content involved and sought to probe

their knowledge about the difficulties that Basic Education students might have when solving such mathematical tasks.

After the first stage, future teachers were divided into SG, from four to six participants to solve the PLTT autonomously. The PLTT were structured in three parts, containing questions for discussion based on mathematical tasks and protocols with student resolution; involving connections with academic mathematics and authentic practices of Basic Education teachers through videos or class reports, which we call "Cases of Practice".

PLTT-1, entitled "Mundo Paralelo (Parallel World, in English)", was made available to the FT accompanied by instructions for the SG to discuss and prepare their resolutions. After sending the FT resolutions, a plenary session was held with all participants, in which the PLTT resolutions were shared, discussed and systematized.

For classes, as well as data collection, the *Moodle* and *TEAMS* platforms were used. The first was used to share teaching materials and send resolutions, while the second enabled video recording. The asynchronous meetings of the SG were held in the technological environment, and the plenary sessions took place in synchronous classes in the first cycle, carried out with a class in the year 2021, and in person in the second cycle, held with a new class in 2022. Figure 3 summarizes the entire process.

Figure 3: PLTT development in each cycle
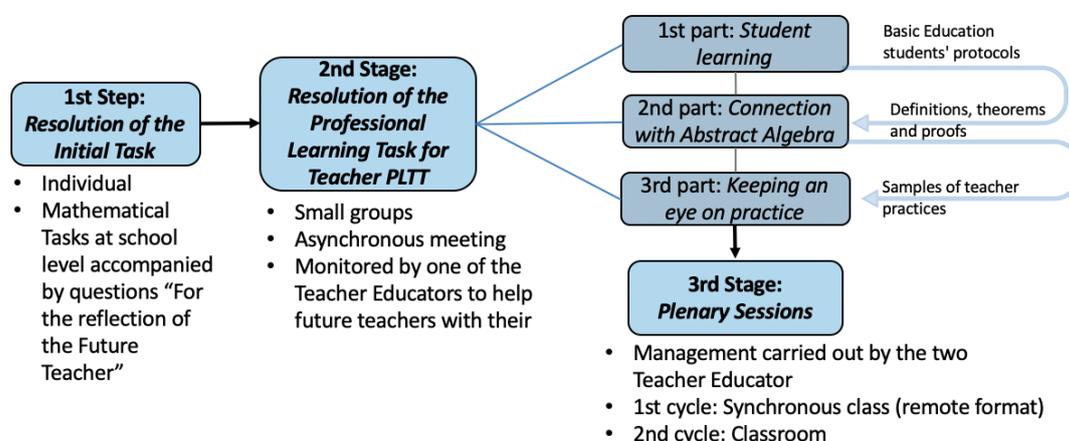
Source: Elaborated by the authors

Likewise, the PLTT-2, entitled "Mundo Antagônico (Antagonistic World, in English)", was made available to the SGs and the second and third stages of the process described in Figure 3 were repeated. After each plenary session, the teacher educators talked about their impressions and established positive and negative points observed, in order to reflect their actions and propose new possibilities for reapplication of PLTT. The audio or video recording of these conversations, together with the evaluations of the FT carried out through a questionnaire, make up the moment of reflection for the teacher educators.

The entire process involving the two PLTTs was carried out with two classes from the same degree course in subsequent years and, therefore, comprises two cycles of Planning, Development and Reflection, or PDR[2] cycle (Trevisan *et al*., 2020).

6 Methods and procedures

Based on the objectives of the study of which this article[3] is a part, a qualitative approach

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[2] We consider that the development of the two TAPs is part of a PDR cycle, as both were developed simultaneously.

[3] This article composes the doctoral thesis elaborated in the Graduate Program in Teaching and History of Science and

was adopted, and the data collected for analysis led us to an interpretative perspective of social constructivism (Esteban, 2010). Design-Based Research (DBR) (Cobb, Confrey, DiSessa, Lehrer & Schauble, 2003) was adopted as a method in order to investigate PLTT as an artifact in approaching a problem. This was achieved through DBR cycles, in which each cycle sought to outline how to use such artifacts, develop plans for their use and, finally, evaluate the results of use and possible repercussions, with a view to executing a new cycle (Barbosa & Oliveira, 2015). Therefore, this article presents part of the evaluation of data from the development phase of the PLTTs in the two PDR cycles, as previously described in the context of the study. It is worth clarifying that throughout the entire development phase of the DBR, the researcher (and first author of this paper) established a partnership with the teacher educator responsible for the FT classes so that, together, they could apply the PLTT with the objective of approaching the ASG with the objective of approaching the ASG with a view to teaching school mathematics content.

The teacher educator "called Paulista" was chosen to participate in the research because she has experience in teaching Algebra for the degree in Mathematics and because she demonstrates an interest in using approaches that involve FT in experiences related to the basic education classroom. In addition, her academic background (degree in Mathematics, Master's in Mathematics, in the area of Algebra, and PhD in Mathematics Education) drew attention to a possible multifaceted vision of the teacher educator.

In total, in addition to the teacher educator, 35 future teachers participated, 15 in the 1st cycle (divided into three SG) and 20 in the 2nd cycle (divided into five SG). Most of these FTs (11 out of 15 FT – 1st cycle; and 14 out of 20 FT – 2nd cycle) reported having some experience with teaching mathematics when starting the subject (e.g. participation in programs such as the Institutional Program for Teaching Initiation Scholarship (PIBID) and Pedagogical Residency), and almost all intended to carry out the supervised internship in that semester, except for two participants in the 2nd cycle.

The data collected and used for analysis in this article consisted of the resolutions produced by the FT and the video recordings, both of the moments of resolution of the PLTT in the SG, as well as of the plenary sessions conducted by the teacher educators in the two PDR cycles. This is illustrated in Table 1, where the $PG3_2$-T1 code, for example, indicates the recording of the small group of FT 3 of the 2nd cycle when solving PLTT-1.

Table 1: Codes for PLTT development video data

Cycle	PLTT	Small Groups – SG					Plenary Sessions
1st cycle	**PLTT-1**	$SG1_1$-T1	$SG2_1$-T1	$SG3_1$-T1	---	---	$P1_1$-T1
	PLTT-2	$SG1_1$-T2	$SG2_1$-T2	$SG3_1$-T2	---	---	$P2_1$-T1
2nd cycle	**PLTT-1**	$SG1_2$-T1	$SG2_2$-T1	$SG3_2$-T1	$SG4_2$-T1	$SG5_2$-T1	$P1_2$-T2
	PLTT-2	$SG1_2$-T2	$SG2_2$-T2	$SG3_2$-T2	$SG4_2$-T2	$SG5_2$-T2	$P2_2$-T2

Source: Elaborated by the authors

Based on the collected data, a descriptive report detailing the entire process was prepared, from which three episodes were selected to elucidate the stages of the process and, thus, compose the corpus of analysis. As a data analysis methodology, considering the research questions, Content Analysis (Bardin, 2016) was used to support the construction of categories.

Mathematics at the Federal University of ABC, organized in multipaper format, written by the first author and supervised by the other authors.

Such construction took place through a deductive approach based on the theoretical contribution previously presented.

The organization of the categories is presented considering two aspects: the first refers to the knowledge mobilized by the FT when they get involved with the PLTT and is theoretically based on the ideas of Dreher *et al*. (2018) on SRCK and is presented in Table 2:

Table 2: Analysis categories related to the knowledge of future teachers

Theoretical Foundations	Category	Description	Indicators
SRCK Knowledge of school-related content (DREHER *et al.*, 2018)	Curriculum Structure and Core Ideas (SR-CSCI)	Meta-knowledge of mathematics and its fundamental ideas.	• Address fundamental concepts and ideas and their meanings in different school years
	From bottom to top, or from SM to AM (SR-SmAm)	Exploration of school mathematics topics in the light of academic mathematics.	• Assess the mathematical integrity of concepts, theorems, proofs or procedures used in school contexts; • Treat a definition in the school context considering Academic Mathematics; • Identify reasons and/or evidence behind implicit statements and assumptions in the school context.
	From top to bottom, or from AM to SM (SR-AmSm)	Transformation of academic mathematics for didactic purposes	• Trimming mathematical ideas for school purposes while maintaining mathematical integrity; • Unpack a mathematical idea; • Evaluate problems/tasks for teaching a certain mathematical idea.

Source: Elaborated by the authors

The second aspect refers to the teacher educators' actions to offer PLO (Ribeiro & Ponte, 2020) and include the use of mathematical knowledge through practices in teaching algebra (McCrory *et al*., 2012) and practices for orchestration of discussions (Borko *et al*., 2014). This is summarized in Table 3:

Table 3: Analysis categories related to the practice of teacher educators.

Theoretical Foundations	Category	Description	Indicators
Practices with the use of mathematical knowledge in teaching (McCRORY *et al*., 2012)	Connecting concepts (PR-CO)	Connection between mathematical concepts and ideas.	• Connect and link mathematical concepts and ideas.
	Trimming ideas (PR-TR)	Recognition of more complex knowledge in favor of a concept	• Targeting academic mathematics to gain local understanding.
	Unpacking (PR-UN)	Evidence for promoting	• Help FTs to understand procedures; • Help FTs to recognize constraints.

		mathematical understanding.	
Essential practices for orchestrating discussions (BORKO *et al.*, 2014)	Awakening (OR-AW)	Stimulus for reflections to be shared	• Request a descriptive comment; • Attract the FT to the discussion.
	Search for evidence (OR-SE)	Search for evidence in the presented claims	• Guide the FT in the elaboration of the exposed ideas; • Draw inferences from the observations presented.
	Connect professional knowledge (OR-CK)	Connection to mathematical and pedagogical analyzes and ideas.	• Request FTs to make connections between mathematical and pedagogical topics. • Systematize the discussions according to the objectives of the class.

Source: Elaborated by the authors

7 Results

For the presentation of our results, the FT had their names identified by the names of neighborhoods or cities close to São Paulo, and the moments of the resolution of the PLTT in SG and plenary sessions are presented through three episodes named as *Knowledge to justify*, *Knowledge to explore* and *Knowledge to connect*.

1nd Episode: Knowledge to justify

In PLTT-1, among the protocols with student resolutions and excerpts from teaching materials, a vignette of a lesson was presented in which the procedure for performing the division between two fractions was presented (Figure 4):

Figure 4: Part of the 3rd part of PLTT-1

1st Practice Case: *A lesson on division of fractions*

During a class intended for the 7th year of Elementary School, the following procedure was presented to carry out the division between two fractions:

Vignette 1:
https://www.youtube.com/watch?v=0FYIjQJZahM&list=PLXyA-zl-y4WETisuhkP18iz8aetBjb2yb&index=25

Video crop 7:44 to 8:43

Knowing that $(Q^*, \cdot)$ is an example of a group, answer the following questions:
1. What does it mean to say, "we invert this operation [multiplication] and to compensate, we have to invert this last fraction"? Indicate the connections with the algebraic structure of the Group presented above.
2. How would you use these links to the group structure to explain the procedure presented in the video to your students?

Source: Search data

Based on a vignette, the teacher educators included two questions in the 3rd part of the PLTT-1, with the aim of *asking the FTs to make connections between mathematical and pedagogical topics.* For this, it was necessary for the FTs to use the ASG to think of a teaching strategy, which would be explored in the first question. During PLTT-1 resolution in the 1st

cycle, SG1 explored other cases (e.g., $\frac{9}{10} : \frac{3}{5}$) to seek a justification and commented:

> Grajaú: *Now, why the division between fractions, I can change the operation, turn it into a multiplication and invert* [the second fraction]. *I don't know how to explain it. I don't know how to justify it. Really, I just know you can.* [...]
>
> Lapa: *I think it's that property of Groups, which is the existence of the multiplicative inverse. But leaving this property and going there for the algorithm is a long way off for me* (SG1$_1$-T1, 2021.

During the resolution, the FTs had difficulties in justifying the procedure presented (*I don't know how to explain it*) and pointed to the distance between school and academic content (*for me it is very distant*). One of the teacher educators monitored the SG during the autonomous work and carried out an intervention to help the FT to connect the division procedure between fractions with the properties of the ASG:

> Researcher: *Look, what she* [the teacher in the video] *is doing is the 'algorithm' that we've learned all our lives, right?* [...] *Why invert the second to switch to multiplication?* [...]
>
> Lapa: *Are we multiplying by 1?*
>
> Researcher: Hmm, there's something out there [...] *Take these fractions there* [referring to the multiplication that the FP that was on the screen, 9/10 divided by 3/5], and instead of writing it that way, what is the other way that we can write this division? On paper it might be easier (SG1$_1$-T1, 2021).

The teacher educator starts her mediation, redoing the question addressed in the PLTT and seeks to rescue the school context experienced by the FTs (*it is the algorithm that we have learned all our lives, right?*), and thus attracts the attention of the FTs to the discussion of the procedure treated there. Next, she confirms the assumption of one of the FTs (there's something out there!) and suggests a new representation for division inserted by them, and thus draw inferences from the presented observations. With its help, FTs are able to establish a justification for the procedure:

> Grajaú: *[...] I thought like this: we go to multiplication. [...] And in the Group property I have the inverse element, which then gives 1 [when operated with the symmetric] [...] so if I multiply by '1', and this '1' is, above and below the fraction, [...] the inverse of the denominator, so I have 1 in the denominator and then I work only with the top* [fractions], *so it will be 9/10 times 5/3.*
>
> Lapa: *Wow, now I understand.*
>
> Grajaú: *Wow!*
>
> Lapa: *The bottom one is the inverse, hence the bottom one 'adds' because it multiplies by the inverse, and it's 1. Multiplying by the inverse is 1, which is the bottom part of the fraction, and the top part you multiply by 5/3, because that's what's left over from this process* (SG1$_1$-T1, 2021).

To justify the procedure presented in PLTT-1, the FT used the properties of the existence of the neutral element (*If I wanted my denominator to be 1*) and symmetrical (the inverse of the denominator) of the Group in question, and thus evaluate the mathematical integrity of the procedure used in the school context. Then, two FTs rewrote the example in response to the second PLTT question and explained to the other SG participants what was discussed (Figure 5).

SG participants used the ASG properties to justify the division between two fractions, demonstrating how they would teach this idea to Basic Education students. This allowed them

to construct a new meaning for the procedure explored in trimming a mathematical idea for a school purpose while maintaining mathematical integrity.

Figure 5: Example presented by the FT in SG1

$$\frac{\frac{9}{10}}{\frac{3}{5}} * 1 = \frac{\frac{9}{10}}{\frac{3}{5}} * \frac{\frac{5}{3}}{\frac{5}{3}} = \frac{\frac{9}{10}}{1} * \frac{\frac{5}{3}}{1} = \frac{9}{10} \cdot \frac{5}{3}$$

Source: Search data

After connecting the academic content with the school content, the SG indicated a possible change in the way they can teach the referred procedure. This can be seen in the following dialog:

> Lapa: *Now it made sense! I think that in the video she* [the teacher] *could have given a similar example, because it is much clearer that way, than just putting the algorithm.*
>
> Grajaú: *[...] we accept it as truth and do it [...] and only now I found out why we can do this* (SG1$_1$-S1, 2021).

The FTs improved their knowledge of mathematical content and reflected on a possibility to teach division between fractions, compared to the way they had learned previously.

In this episode, the teacher educator played a fundamental role in the use of orchestration practices (OR-AW; OR-SE; OR-CK) to help the FP to understand the procedure for dividing between fractions, which is characterized as a practice of unpacking knowledge to justify a procedure (PR-UN). In addition, the teacher educator directed academic mathematics to obtain a local understanding, which characterizes a practice of trimming (PR-TR).

On the other hand, the FT explored the knowledge related to the ASG linked to the teaching of operations with rationals, starting sometimes from the school context, sometimes from the algebraic properties (SR-SmAm, SR-AmSm).

2nd Episode: Knowledge to explore

Mathematical task 2, which was part of PLTT-1, addressed the content of matrix multiplication and was also accompanied by protocols from Basic Education students, part of which are shown in Figure 6.

Figure 6: Mathematical Task 2 accompanied by one of the protocols

Math Task 2

Cilene has some doubts regarding the multiplication of two matrices (A and B).

a) Can the product A.B, for example, exist and the product B.A not exist?

b) Can the product matrices A.B and B.A be different types?

c) Can we have A.B = B.A and A.B ≠ B.A?

Protocol of a student for item c)

c) Sim, se A for matriz identidade, por exemplo, A.B = B.A O (mes) mesmo vale se B for matriz identidade.
Se nem A nem B forem identidade, teremos A.B ≠ B.A

Translate: c)Yes, if A is an identity matrix, for example, A.B = B.A
The same holds true if B is an identity matrix.
If neither A nor B is identity, then A.B ≠ B.A

Source: Search data

Still in the first cycle, during the plenary session, the teacher educator called Paulista asks the FT about item c (Figure 6):

Paulista: *What kind of property are they asking for here?*

Mooca: *The Commutative?*

Paulista: *Yes! What do you teach or learn in basic education [...], what happens to the matrices? [...] what may or may not occur* [referring to commutativity]

Mooca: *No because we learn that the commutative is not valid for them* (P1_1-T1).

Paulista seeks to attract FTs to the discussion and, at the same time, rescue the common knowledge acquired by them about commutative property during Basic Education. Then, she directs the discussion to a deeper understanding of the topic:

Paulista: *The way we decorate [...] that the set of matrices for the multiplication operation is not commutative. There! Then the question is: for every type of matrix? [...] Well, there are some restrictions for us to multiply matrices, but could it be that within this set there are no matrices that when I multiply A by B and B by A, these products would not be equal? Is there?*

Capão Redondo: *There is.*

Paulista: *Which are they? Give me an example.*

Capão Redondo: *An example would be A equals B, and they are square.* (P1_1-T1).

The teacher educators directed questions to make the FT identify reasons behind implicit statements and assumptions in the school context, addressed through restrictions. This exercise is commonly done in disciplines such as Algebra, in which examples of mathematical objects already known are used to restrict or expand them in order to explore new ideas or concepts objectified by the discipline.

In the 2nd part of PLTT, the definitions of ASG and Abelian Group are presented, and the teacher educators explore such properties from an academic point of view, emphasizing the need for a set and a defined operation. And she continues to explore the commutative property:

Paulista: *This is an example that the statement* [of the Mathematics task], *according to you, has to be very clear. I'm thinking in the space of all matrices? Can I reduce this space, can I work with some types of specific elements of matrices that are worth this property [commutativity]? [...]*

Researcher: *Let's restrict it to 2x2. So we already have the identity matrix* [referring to the neutral element]. *When can I guarantee that the elements of matrices of order 2x2 have an inverse? [...]*

Bela Vista: *Determinant?!*

Researcher: *Yes! I need the determinant to be...*

Bela Vista: *Be... different from zero!*

Researcher: *Yes. [...] Who systematizes for me, an answer to the second question* [from PLTT]: *'Which set of matrices, through the multiplication operation, presents the properties of a Group?' I'm going to give a kick: order 2, but it will be valid for order n.*

Capão Redondo: *Any matrix A that has a non-zero determinant.*

Researcher: *That, and will it be commutative? Will it be an abelian group?*

Capão Redondo: *No!*

Researcher: *Only if I make another restriction, which was what we discussed back there* [referring to the

discussion of mathematical task 2] (P1$_1$-T1).

The teacher educators explore the existence of sets of matrices in which the algebraic properties of multiplication, such as commutativity and the existence of the symmetric element, can be present when restricting and extending certain sets with an operation. In this way, the trainers help the FT to recognize restrictions linked to the idea of commutativity of multiplication in sets of matrices in order to have an ASG.

By using examples of subsets of matrices that satisfy the commutativity property of multiplication, teacher educators help FTs to assess the mathematical integrity of the concept. For this, they seek to extract inferences from the observations presented through examples in which matrix multiplication is commutative, as the FT identified only trivial cases in the Initial Task, such as the use of the identity matrix or null or even when matrices A and B are equal. The trainers used matrices of order 2 with a determinant other than 0 and non-zero diagonal matrices to exemplify a Commutative Group and allow the FTs to evaluate the teaching task that was initially presented to them.

The teacher educators returned to the question of PLTT to systematize the discussions according to the objective of the class, which was to explore subsets of matrices that present a subgroup structure. She asked the participation of the FT to answer the question: *Which set of matrices, through the multiplication operation, presents the properties of a Group?"*

The teacher educators used the practice of unpacking (PR-UN) to address the commutativity (or non) of mathematical objects, using matrices as an example, and showed how to build sets in which multiplication is commutative or not, while orchestrating the discussions (OR -AW; OR-SE; OR-CK). These discussions were connected with a mathematical task as a backdrop and helped the FTs to use school and academic knowledge together (SR-SmAm; SrAmSm).

3rd Episode: Knowledge to connect

PLTT-2 "Antagonistic World" involved three Mathematical Tasks related to the concept of functions at different school levels. The episode in focus at this moment refers to one of these Mathematical Tasks, shown in Figure 7.

Figure 7: PLTT-2 Mathematical Task 3 accompanied by a protocol

Math Task 3: (Adapted from UESC)
A message can be encoded in a number of ways. If, for example, a positive integer n is associated with each letter of the alphabet, considering a function f(n), known only to the sender and recipient of the message, it is possible to establish a form of codification. In this case, function f is used to encode and its inverse f^{-1} to decode the message. Considering A = 1, B= 2, ... W = 23, X= 24, Y = 25, Z = 26 and f(n) = n + 3 to encode the letter U, instead of transmitting the number associated with it, which is 32, we transmit the letter associated with f(32) = 24, which is X. To decode the letter X received, we observe that it corresponds to 24. Therefore, f^{-1} (24) = 21, which is U. Assuming, hypothetically, that the function f(x) = 2x+3 can be considered a key function for encoding a certain pattern of messages, what is the expression of its inverse to be used in decoding these messages?

A student's protocol for TM3
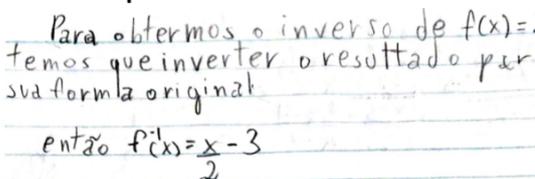
Para obtermos o inverso de f(x)=. temos que inverter o resultado por sua formla original

então f^{-1}(x)= $\frac{x-3}{2}$

Translate: To get the inverse of f(x) = 2x+3 we have to invert the result to its original form

so f^{-1}(x) = $\frac{x}{2}$ - 3

Source: Research Data

During the plenary of the second cycle of PLTT-2 (P2$_2$-T2), the teacher educator Paulista compared the symmetrical elements of the multiplication of rational numbers, addressed in PLTT-1, with the inverse functions presented in PLTT-2, based on an observation of an FT over subscript use -1. She also presented an interpretation for the mistake presented in the protocol (Figure 7):

Diadema: This is a problem with using the same symbol to represent different things [...] you have to establish these things; you have to make it clear!

Paulista: The '-1' is associated with what? *When I say x to the -1, what are you thinking about?*

Some FT: *In the multiplicative inverse.*

Paulista: *In the multiplicative inverse, right? Of a number!* [...] *So, when I ask you, "what is the multiplicative inverse of 2?", "is the one half" why?*

Perus: *Because if you multiply 2 by ½ it gives the neutral element.*

Paulista: Which is 1. *We don't say that to the student. You just teach the student like this: "ah, it's the multiplicative inverse of 2, switch!"* [numerator by denominator]. [...] *If it's 2 then it's ½. But why just change?* [...] *You change why? Look, it's the inverse because when I take it I take ½ and multiple by 2 gives 1, which is the neutral element.* And then he [the student] knows what he is doing [...] because every time he sees the '-1', he switches". Then he sees the function, and "exchange" [...]. *And actually that's not the concept of the symmetrical element.* The symmetric is always when you operate [an element] with the symmetric and get the neutral element [...] *And here, when we were talking, the student thought that way.* [...] he only inverted the 2, because he only learned to invert 'numbers'.

Paulista: *First, the inverse of a function is one thing, and the multiplicative inverse is another. The concept is the same: if it is inverse I multiplied and it gives the neutral element, but what is the operation here?* (P2$_2$-T2, 2022).

Diadema started the discussion about the use of the subscript -1, without mentioning the specific contents and which could identify aspects of the concepts in different school years for the use of that notation. The teacher educator used questions to attract the FT to the discussion and some presented an interpretation of the subscript -1 as an indicator for the multiplicative inverse. Although they used the term "multiplicative inverse", linked to academic knowledge, the FT had doubts about this common concept. The teacher educator explored the definition of a symmetrical element to link the concept of functions to the use of the subscript -1 with comments and questions to guide the FT in the elaboration of the exposed ideas.

FT Perus mentioned the relationship between element, its symmetrical and neutral and, thus, treated a definition in the school context considering Academic Mathematics, and Paulista explored this concept, by pointing out the definition of symmetrical element. In this way, the trainer connected and linked mathematical concepts and ideas. Next, they sought to understand the meaning of the inverse function:

Paulista: First, the inverse of a function is one thing, and the multiplicative inverse [of a number] is another. The concept is the same: if it is inverse [symmetric] I operate and it gives the neutral element, but what is the operation here? [a few seconds of silence]

Researcher: Why is this function [f(x)=2x+3] the inverse of this [f(x)= (x+3)/2]? *Gotta have the operation! What is the operation?* [...]

Some FT: *Composition!*

Other FT: *Multiplication?*

Researcher: *So can I use multiplication?* [...] When you found this inverse function, what procedure did you use?

São Miguel: *We reversed the variables!* [...]

Researcher: *You used a procedure, but that procedure is linked to the composition, right? Why?* [FTs reflect and comment among themselves].

Researcher: *The relationship between the two functions is composition, because one is inverse of the other because of composition. And who is the neutral element of the composition?*

São Miguel: *x.*

Diadema: *The identity function*

Researcher: *Yes, the identity function, I(x)=x.*

Paulista: *Because now it's the composition!* (P2$_2$-T2, 2022).

There was a lack of consensus among the FTs regarding the operation involved in the Mathematical Task (composition or multiplication), which was explored by the trainer when requesting a descriptive comment asking, "which procedure did you use?" to rescue what was done by the FT when solving the Mathematical Task and discussed in the SG. As the FT did not feel confident in exposing their assumptions, the trainer used new questions to draw inferences from the observations presented and attract the FT to the discussion in the search for the objectified connections.

Researcher: *And what is the relationship of the identity function to these functions here?* [points to $f(x) = 2x + 3$ and $f(x) = (x + 3)/2$ written on the blackboard].

Diadema: When you operate one [function] with the other, you give the function identity.

Researcher: *Yes! If I do $f \circ f^{-1}$ or $f^{-1} \circ f$ results in the identity function.* [Writes on the board $f \circ f^{-1} = f^{-1} \circ f = id$] (P2$_2$-T2, 2022).

Guided by such questions, the FT were able to unpack a mathematical idea based on the relationship between an element, its symmetric and the neutral element of the functions in relation to the composition operation ($f \circ f^{-1} = f^{-1} \circ f = id$). This idea was previously discussed with the multiplication of rational numbers, as exemplified in the speech by Perus ($2 \cdot \frac{1}{2} = 1$), where both involve the use of the subscript -1, initially raised by Diadema. Such statements led the teacher educators to systematize the discussions according to the objective of the class, which was to explore the Group of functions in relation to composition.

With this process, the FT identified a reason for the common use of the subscript -1 in apparently different concepts, when addressing fundamental concepts and ideas and their meanings in different school years, leading them to identify a reason that was behind a statement and implicit assumptions in the school context.

In this episode, the teacher educators used practices to orchestrate discussions (OR-AW; OR-SE; OR-CK) and connect inverse concepts, used in Basic Education with the definition of symmetric in an ASG (PR-CO). The FT, on the other hand, mobilize knowledge about the use of the subscript -1 in the school curriculum and present points of view of school and academic mathematics (SR-CSCI; SR-SmAm; SR-AmSm), which can lead to a new meaning of the mathematical concept for the FTs.

8 Discussion

In order to answer the research questions: *How are professional knowledge modified by future teachers in carrying out Professional Learning Tasks in an Exploratory Teaching*

approach? and *What practices and how do trainers use them to generate professional learning opportunities when they seek to articulate mathematical knowledge of the algebraic structure of Groups to the teaching of school contents?*, we discuss the results in the light of the theoretical framework presented above and present, at the end, a scheme that seeks to synthesize the knowledge mobilized by future teachers and the actions taken by teacher educators in promoting professional learning opportunities.

The Mathematical Tasks made it possible to raise the FT's prior mathematical knowledge, without any mention of academic mathematics or teaching. However, while Records of Practice, such as protocols and vignettes, were explored by teacher educators through Exploratory Teaching linked to practices for orchestrating mathematical and didactical discussions (Canavarro *et al*., 2012; Borko *et al*., 2014), moments were observed when professional knowledge was addressed to provide opportunities for professional learning of FTs, (Ribeiro & Ponte, 2019, 2020; Aguiar *et al*., 2021b).

In the first episode, knowledge of academic and school mathematics was mobilized by the FP to justify the division procedure between fractions based on the properties of the ASG (Zazkis & Marmur, 2018). The FT were able to use the ideas discussed to think about how to teach the procedure in a different way than that initially presented in the discussions, which indicates a change in the way of teaching, that is, in the SCK. In this episode, the teacher educators' direction to trim and unpack mathematical knowledge was essential for the mobilization of professional knowledge of the FTs (McCrory *et al*., 2012).

In the second episode discussed the matrix multiplication operation and sought to find sets in which this operation is commutative (Zazkis & Marmur, 2018). The direction taken by the teacher educators allowed the unpacking of the idea (McCrory *et al*., 2012) related to the non-commutativity of multiplication between matrices and the existence of subsets in which this property can be contemplated (Wasserman, 2017). Although this has not been addressed as such knowledge could change the way FTs can teach, there is a change in their understanding in relation to the commutative property, which is defended by Wasserman (2016).

In the last episode, the elements of PLTT allowed the FT to problematize the use of the subscript -1 linked to the relationship between the elements of a Group G and the neutral element of the operation (Zazkis & Kontorovich, 2016; Zazkis & Marmur, 2018; Wasserman, 2016). Exploring different Groups allowed the educators to reveal their understanding of the use of the subscript -1 in different school contents that run through the curriculum (McCrory *et al*., 2012; Dreher *et al*., 2018).

In the three episodes, the teacher educators used the essential practices for Orchestrating Discussions (Borko *et al*., 2014), and the first episode was based on the use of a video. In the course of Algebra in the undergraduate mathematics course, the teacher educators connected knowledge about ASG with school mathematics concepts such as rationales, matrices and functions (Zazkis & Marmur, 2018), using the practices of connecting, trimming and unpacking knowledge (McCrory, *et al*., 2012; Gonçalves *et al*., 2022) which was characterized as PLO to FT (Ribeiro & Ponte, 2019, 2020) as the PLTT were discussed in an environment for this purpose (Ticknor, 2012).

The FT had previous knowledge about procedures and concepts (CCK), but as they became involved with the PLTT and were asked in the different phases of Exploratory Teaching (Canavarro *et al*., 2012), they mobilized content knowledge related to the school, the SRCK (Dreher *et al*., 2018), obtaining a better understanding of the mathematical concepts addressed, as shown in Figure 8.

In the end, their knowledge is modified, and these can be used in favor of teaching, which would be linked to SCK. This was made explicit in the first episode when it is proposed to teach the procedure based on the algebraic structure discussed.

Figure 8: Relationship between content knowledge constructs
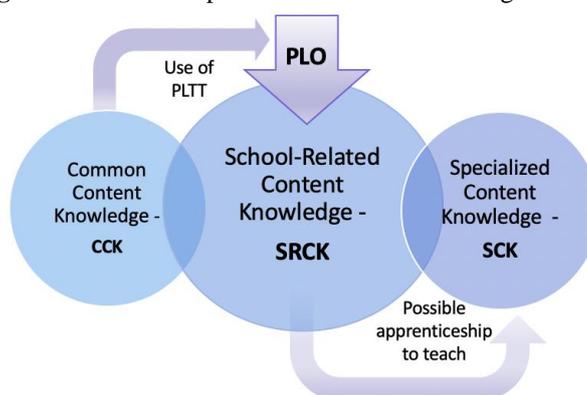
Source: Prepared by the authors

9 Conclusions

Using the connections between school and academic mathematics, in initial teacher education courses, is still a challenge given the lack of research on the subject (Moreira & David, 2008; Wasserman, 2017). In line with this, this article sought to understand how professional learning tasks are carried out in classes of an Algebra discipline in the Mathematics degree, in order to thus, reveal a path through which teacher educators can use these connections in the teaching of school contents.

The use of the SRCK construct to connect school and academic mathematics in and for teacher professional learning seems to be essential in a context of prospective teacher education for mathematics teachers for Basic Education (Moreira & David, 2008; Speer *et al*., 2015), as it allows exploring the necessary facets for knowledge of the content and its use in teaching (Dreher *et al*., 2018).

In addition, the exploration of Mathematical Knowledge from Algebra to Teaching expands what was exposed by McCrory *et al*. (2012), since practices related to this knowledge were observed in an academic teaching environment. As for the use of PLTT associated with practices for orchestration, such articulation seems to us a way to promote mathematical and didactical discussions in an initial educational context (Borko *et al*., 2014; Ribeiro & Ponte, 2020, Trevisan *et al*., 2023).

However, it is important to emphasize that the modification of knowledge was not uniform among the participants and, therefore, it is still necessary to investigate whether and how this learning is incorporated into teaching practice, in addition to new studies to understand how the discussions among the participants contributed for the creation of the PLO during this process. Finally, we understand that more research is also needed to explore the possibilities of addressing the connections between academic and school mathematics in other areas of mathematics, such as Analysis or Geometry, to enrich the prospective mathematics teachers education (Moreira & David, 2008).

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