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Don’t memorize theorem proofs! The Modern Geometry of Osvaldo Sangiorgi

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Abstract: This paper aims to examine how deductive geometry was constituted in two didactic book collections by Osvaldo Sangiorgi — named by premodern (1950s) and the modern (1960s) ones. The guiding question of the study is: how does Sangiorgi change the proposal of a deductive geometry for the 3rd grade of junior Brazilian high schools in the modern collection compared to the premodern collection? Postulates, theorems, and proofs are examined in detail, emphasizing the quantitative and qualitative aspects, as well as the methodological recommendations. This analyzes shows that the modern collection brought about significant changes and contributions both in the scope of Euclidean geometry, of a geometry to teach, and in the methodological didactic aspects by proposing the insertion of exploratory and experimental exercises, different registers of representations, which means, in our point of view, a geometry for teaching, which can be interpreted as an intuitive geometry, taken from the 1st and 2nd grades of the Brazilian Minimum Program of 1951.

Keywords: Deductive Geometry. Textbook. Modern Mathematics Movement. Junior High School.

¡No memorices las demostraciones de teoremas! La Geometría Moderna de Osvaldo Sangiorgi

***Resumen**:* El artículo tiene como objetivo examinar cómo se constituyó la geometría deductiva en dos colecciones didácticas de Osvaldo Sangiorgi — la premoderna (década de 1950) y la moderna (década de 1960). La pregunta orientadora del estudio es: ¿cómo cambia Sangiorgi la propuesta de una geometría deductiva para el 3º año de secundaria en la colección moderna en comparación con la colección premoderna? Se examinan en detalle postulados, teoremas y demostraciones, enfatizando aspectos cuantitativos y cualitativos, así como recomendaciones metodológicas. Los análisis revelan que la colección moderna trajo alteraciones significativas tanto en el ámbito de la geometría euclidiana, de una geometría a enseñar, como en los aspectos didácticos metodológicos al proponer la inserción de ejercicios exploratorios y experimentales, diferentes registros de representaciones, lo que corresponde, en nuestro punto de vista, al punto de vista, a una geometría para enseñar, que se puede interpretar como una geometría intuitiva, tomada de los grados 1° y 2° del Programa Mínimo de 1951.

***Palabras clave**:* Geometría Deductiva. Libro de Texto. Movimiento Matemático Moderno. Escuela Secundaria.

Não decore demonstrações de teoremas! A Geometria Moderna de Osvaldo Sangiorgi

Resumo: O artigo tem por objetivo examinar como se constituiu a geometria dedutiva em duas coleções didáticas de Osvaldo Sangiorgi — a pré-moderna (década de 1950) e a moderna (década de 1960). A pergunta norteadora do estudo é: como Sangiorgi altera a proposta de uma geometria dedutiva para a 3ª série ginasial na coleção moderna comparativamente à coleção pré-moderna? Examina-se detalhadamente os postulados, os teoremas e as demonstrações, dando ênfase aos aspectos quantitativos e qualitativos, bem como às recomendações metodológicas. As análises revelam que a coleção moderna trouxe alterações significativas tanto no âmbito da geometria euclidiana, de uma *geometria a ensinar*, como nos aspectos didáticos metodológicos ao propor a inserção de exercícios exploratórios e experimentais, diferentes registros de representações, o que corresponde, em nosso ponto de vista, a uma *geometria para ensinar*, que pode ser interpretada como uma geometria intuitiva, retirada das 1ª e 2ª séries no Programa Mínimo de 1951.

Palavras-chave: Geometria Dedutiva. Livro Didático. Movimento da Matemática Moderna. Ensino Ginasial.

1 Preliminary considerations

The theme of the proposals for teaching geometry during the period of the Modern Mathematics Movement[1] (MMM) is controversial, considering the complexity of the movement, of international scope, with different perspectives and proposals on how to develop a plausible deductive geometry for students of the old Brazilian middle school, formerly designated as "*curso ginasial*"[2], during the 1950s and 1960s, as pointed out by the study by Leme da Silva (2022). In addition to the complexity of the MMM, we cannot disregard, in the task of representing the past of geometry teaching, the specificity of our country, which, due to its continental dimensions, followed different guidelines depending on the state and the region.

Given the above and specific representations of the past, which tend to emphasize negative aspects related to the teaching of geometry and MMM[3], we consider it pertinent to resume investigations of a historical nature on the teaching of geometry during that movement[4]. However, in this new challenge, the object of study will focus on specific knowledge[5], particularly those that the completed studies have already identified as significant changes in Brazilian school culture.

Investigating a particular knowledge also makes it possible to mobilize the theoretical framework of the Swiss group coordinated by Hofstetter, in particular, the concepts of *knowledge to teach* and *knowledge for teaching*. Valente and Bertini (2022) adapted the terms to *mathematics to teach* and *mathematics for teaching*, analyzing them in the context of the relationships produced in school culture to elaborate on a theoretical research object called *mathematics of teaching*. In our study, we consider *geometry to teach* and *geometry for teaching*

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[1] Since the 1950s, commissions, study groups and seminars have been created to discuss proposals for change in mathematics teaching, both in Europe and in America. The proposals defend the unification of the various fields of mathematics, bringing the teaching carried out in basic education closer to that of the university, which corresponds to the language and structure employed by the mathematicians of the time.
[2] Today the middle school course corresponds to the four final years of the primary education, from 6th to 9th grade.
[3] Pavanello (1993) and Caldatto and Pavanello (2015) are cited.
[4] The second author participated in the CAPES/GRICES International Cooperation Project entitled *A Matemática Moderna nas escolas do Brasil e de Portugal: estudos históricos comparativos* [Modern Mathematics in schools in Brazil and Portugal: comparative historical studies] and specifically investigated the teaching of geometry.
[5] The project *História da geometria do ensino e o MMM* [History of teaching geometry and the MMM], in which the two authors participate, submitted to FAPESP, aims to investigate specific geometric knowledge since the 1950s.

as interconnected constituents for the configuration of the *geometry of teaching* during MMM.

Another need identified in the investigative process was knowing about the regulations before MMM. Thus, we conducted a preliminary study[6] on geometry teaching in Francisco Campos (1931), Gustavo Capanema (1942), and Simões Filho (1951) Reforms, summarized in Chart 1.

Chart 1: The geometry teaching in the prescribed curricula (1931-1951)

Reforms	Organization of geometry	Instructions/Guidelines
Campos (1931)	*Geometric Initiation*: 1st and 2nd grades *Geometry*: 3rd, 4th, and 5th grades	▪ Propaedeutic course in geometry ▪ From intuition to gradually achieving formal exposure: from experimentation and sensory perception to analytical reasoning
Capanema (1942)	*Intuitive geometry*: 1st and 2nd grades *Deductive geometry*: 3rd and 4th grades	▪ Intuitive geometry as a smooth transition between experiences with shapes and deductive conception of geometry
Simões Filho (1951)	*Measures*: 1st grade *Geometry*: 3rd and 4th grades	▪ Do not dispense with the appeal to intuition ▪ Gradually awaken the feeling of the need for justification and proof

Source: Adapted by the authors of Jahn and Magalhães (2023)

Chart 1 indicates that deductive geometry is the one that remained in the three norms, and, in 1951, it is exclusive and restricted to the 3rd and 4th grades. We can say that the MMM found the striking presence of deductive geometry in Brazilian middle schools. Thus, we were instigated to understand how the knowledge that translates the normative proposal, that is, the theorems and their proofs, was interpreted and proposed in the 1950s textbooks and from the1960s, with the insertion of modernizing ideas.

In this context, this article aims to examine how deductive geometry was constituted in the two didactic collections of Osvaldo Sangiorgi (OS) — the first, called premodern, referring to the 1950s, and the second, modern, launched in the 1960s — both published by Companhia Editora Nacional, in São Paulo. The collections are considered true bestsellers in number of editions, and, in this sense, the contexts of the didactic productions and of the author certainly need to be explained so that we can understand the success of the collections.

The choice of textbooks as a source of research to know and understand the history of school geometry is justified by being one of the central elements in the school's pedagogical practices, as an interpreter of prescriptive norms, as indicated by Munakata (2016, p. 122),

> [the] notion of school culture refers not only to norms and rules, explicit or not, symbols and representations, in addition to the prescribed knowledge, but also, and above all, to practices, appropriations, attributions of new meanings, resistances, which produces multiple and varied configurations, which typically occur in school. [...] One of those things peculiar to school is precisely the textbook. Surely it may be elsewhere, as in the library of an eccentric collector, in the offices of the appraiser or

[6] The partial results can be read in Pastor and Leme da Silva (2023) and Jahn and Magalhães (2023).

the researcher, but its existence is only justified in and by the school.

It is worth remembering that, in the 1950s, the state of São Paulo witnessed growth in several dimensions: populational, social and economic, becoming the largest Latin American industrial center. In the educational field, there is a significant increase in the school population — in 1940, the network of state middle schools was formed by 37 establishments in the countryside and 3 in the capital, boosting, in 1950, to 143 middle schools in the countryside and 12 in the capital —, accompanied by an increase in the production of textbooks (Valente, 2008).

On the other hand, the academic education of Osvaldo Sangiorgi (1921-2017)[7] included a Degree in Mathematical Sciences in 1941, taken at the Faculdade de Filosofia Ciências e Letras, Education Section of the Universidade de São Paulo (USP), which meant a reference for the secondary course of the time. In addition, Valente (2008) points out that the children of the São Paulo elite, educated in the 1950s, had private classes and preparatory courses with the best teachers. Osvaldo Sangiorgi was "one of those excellent teachers, disputed by weight of gold by the wealthy families of São Paulo" (Valente, 2008, p. 16). In summary, OS was acclaimed due to his education at USP and his performance as a teacher, decisive elements in producing textbooks with massive circulation.

In this article[8], we first look at the premodern collection of textbooks for middle school, a period that preceded the MMM:

> It will be the *Matemática – curso ginasial* [Mathematics – middle school] one of the publisher's bestsellers, launched in 1953. In February of that year, the collection volume destined for the 1st. middle school series is published, with an exact circulation of 20,213 copies. In July of the same year, with a circulation of 20,216 copies, and in November, with a circulation of 25,266, the volumes for the 2nd and 3rd grades come out, respectively. Apparently, the reception of the collection was particularly good since, at the end of 1953, there is a reprint of the first volume: there are 20,167 more textbooks to be used in the first middle school series, according to the "Mapa de Edições" [Map of Editions] of the Companhia Editora Nacional (Valente, 2008, p. 19).

In the chapter entitled *Osvaldo Sangiorgi, um bestseller*, Valente (2008) presents many data, arguments, and justifications to convince us of the *bestseller* attribute assigned to teacher Sangiorgi's textbook. He was already a recognized author at Companhia Editora Nacional and was an assiduous collaborator of the journal *Atualidades Pedagógicas*, of the same publisher, between 1954 and 1960, criticizing the 1931 Francisco Campos Reform, the 1942 Capanema Reform, and the 1951 Simões Filho Reform. Sangiorgi also used the Congressos de Ensino de Matemática [Mathematics Teaching Congresses][9] to discuss the middle school teaching, bringing reflexes to the premodern collection. At the first congress, held in Salvador/BA in 1955, he put on the agenda the Programa Mínimo [Minimum Program] of 1951, of the so-called Simões Filho Reform and its unenforceability with a workload of three hours per week for the mathematics teaching. All these factors corroborate the increase in the circulation of the

[7] A more detailed study on Osvaldo Sangiorgi and the MMM can be read in the textbook *Osvaldo Sangiorgi: a modern teacher* (Valente, 2008).
[8] This article is an excerpt from post-doctoral research, developed by the first author under the supervision of the second author, in the Postgraduate Program in Mathematics Education at Universidade Estadual Paulista.
[9] The I Congress took place in 1955, in Bahia; the II in Rio Grande do Sul, in 1957; the III in Rio de Janeiro, in 1959; the IV in Pará, in 1962; and the V in São Paulo, in 1966.

editions; in 1957, the first volume reached the mark of 100,000 copies, remaining until 1963, when the 134th edition of the textbook was published.

Sangiorgi closed the 1950s with his collection of textbooks for middle school as a *bestseller* of the Companhia Editora Nacional, carrying forward the debate on mathematics teaching in various spheres — São Paulo education bodies, congresses, examination boards for hiring teachers and boards for students' entrance exams, while criticizing the federal determinations based on Colégio Pedro II and the 1951 Programa Mínimo (Minimum Program). Parallel to the OS's movements, in 1958, new determinations made the elaboration of mathematics teaching programs more flexible, loosening the centralization of Brazilian secondary education (Valente, 2008).

In addition to its articulation at the national level, OS went to the USA after being granted the *Pan American Union* and *National Science Foundation* scholarships for an internship at the University of Kansas between July and August 1960. When he returns to Brazil, he immediately articulates changes in the mathematics program based on his experience abroad. The scenario for the entrance of Modern Mathematics was built through the textbooks for middle school. Valente (2008) states that, in 1963, for use in the 1964 school year, the Companhia Editora Nacional released 240,000 copies of volume 1 of *Matemática – Curso moderno* [Mathematics – Modern course].

This study compared Sangiorgi's modern collection, published in 1964,[10] with the 1950s[11] collection, considering that the new collection presented significant changes in the organization and proposal of mathematics teaching. An innovative character in the modern collection was the publication of *GUIA para uso dos PROFESSORES* [A guide for teachers' use], one for each volume. Valente (2008), once again, maintains in his arguments that OS remains in the position of the bestseller of the Companhia Editora Nacional, with the collection *Matemática – Curso Moderno*:

> The success of the work was confirmed by the new editions of the first volume: in 1965, more than 250 thousand new textbooks are published, and thus, annually, the textbook has print runs in the 250 thousand copies, until 1967, reaching its 10th edition, as attested by the "Mapa de Edições" of the publisher (Valente, 2008, p. 31).

We believe we have, based on the thorough historical study carried out by Valente (2008), adequately justified the prominent role that the two OS's collections had. It is worth reaffirming that we are analyzing the two collections exclusively in the 1950s and 1960s, a period that precedes the changes in educational legislation, starting in the 1970s, especially after Law N. 5692/71[12].

Agreeing with the historian Marc Bloch (2001) when stating that history is not the science of the past but the science of men in time, we believe it is relevant to understand the education and professional trajectory of the man who produced the didactic collections under analysis, in their due time and space.

2 Brief bibliographic review

Santos' dissertation (2023) carried out a hermeneutic study on mathematical proofs in

[10] We analyzed the 3rd Edition, published in 1967.
[11] We analyzed the 35th Edition, published in 1958.
[12] Law N. 5692/71 proposed a new structure for basic education, which became: elementary education, with eight years of schooling, and secondary or high school education, with three years.

MMM, seeking meanings for mathematical proof throughout the history of mathematics. Regarding the proofs in the pedagogical proposals for the classroom, the study indicated that teachers should employ the deductive approach, adopting detailed and rigorous language and that its use was always related to the study of geometry.

In our study, the object of investigation is precisely the deductive geometry prescribed from the 8th grade (middle school), in which the proofs are a central element. Thus, the intention is to conduct a comparative study between the two collections by OS regarding deductive geometry and, more specifically, regarding proofs, to identify the appropriation made by the author about modernizing proposals.

Leme da Silva (2008) conducted a first study on geometry teaching, broadly, on the two works, in which the categories examined were: preface, index, concepts of geometry and deductive geometry. As the main results, the author pointed out that the modern OS's approach came closer to the international trend used in the USA based on Birkhoff than to the proposal of teaching geometry by geometric transformations linked to Klein (whose transformations, in the case of the modern textbook, are presented in the appendix). He also pointed out that OS did not explicitly take a stand on the proposals for teaching geometry. Instead, he incorporated the two trends:

> We can say that Sangiorgi does not take a radical position. He does not abandon Euclidean or deductive geometry but adds new postulates, an exploratory geometry. Regarding the methodological approach, he carries out a more careful development of the concepts and geometric properties. In our view, this change, together with the attempt to recover exploratory aspects, represents a meaningful change in geometry teaching. (Leme da Silva, 2008, p. 91-92)

Búrigo (2015), commenting on the geometry teaching debated at the II Congresso Nacional de Ensino de Matemática [II National Congress of Mathematics Teaching], held in 1957 in Porto Alegre, transcribes part of the testimony of Professor Antonio Rodrigues, founder of the mathematics course and geometry lecturer at the Philosophy College at the Universidade Federal do Rio Grande do Sul (UFRGS), on the pedagogical practices of the period:

> The lack of initial logical concatenation of the theorems and the intuitive character of most of them produce the harmful impression in the student's mind that the proofs constitute teachers' juggling. Taken out of nowhere, they are left loose in the air, without a determined end.
>
> [...] With the accumulation of the study material, the student loses the overview. At this point, they do not realize the mutual relations between the various theorems; at best, they know that this formal proof is supported by the preceding theorem examined by the teacher. They also do not form a clear idea of a theory or have no theory at all. The only thing left for them now is to use memorization to keep the theorems and their proofs during the short term of the exams... (Congresso..., 1959 *apud* Búrigo, 2015, p. 08)

Thus, we can see that although the proofs of theorems appear in the normative recommendations in the 1950s and 1960s and, consequently, are present in the textbooks, the teacher's report indicates that, in practice, students memorized such theorems. In the preface of the premodern collection, OS states his objective regarding geometry teaching: "Great responsibility in initiating high school students into deductive geometry, using a technique that is as demonstrative, accessible, and uniform as possible." (Sangiorgi, 1958, p. 17). In his modern collection, addressing the students in the preface, OS states that

> geometric figures, when treated *rationally*, are a great stimulus for the *deduction* of particular properties common to them, and that could never be accepted if *we just observed*. And if *deduction* is one of the main qualities of being rational, the study of geometry will make it even more rational!" (Sangiorgi, 1967, p. XV, emphasis added)

In this investigation, we seek to understand in more depth how OS mobilizes contributions from MMM so that students "do not memorize the proofs of theorems!" (Sangiorgi, 1966, p. 52), i.e., it seems to be one of the mathematician's main objectives in the modern textbook, since deduction was explicitly present in the normative, both before and at the time of publication of the modern collection. The question we seek to answer is: How does OS change the proposal of a deductive geometry for the seventh grade in the modern collection compared to the premodern collection?

Seeking to answer it, we chose to undertake an analysis of *the geometry to teach* proposal, considered as knowledge of the object of its work, produced by the disciplinary field of mathematics, in this case, Euclidean geometry, as well as *geometry for teaching*, taken as mobilized knowledge as tools for teaching practice, knowledge to teach geometry. We articulated the examination of the deductive exposition throughout the geometry chapters in both collections, emphasizing the aspects related to both the quantity and content of postulates and theorems, as well as the methodological recommendations of the author to implement the proposed teaching.

3 An initial quantitative analysis

Our first comparison was regarding the number of pages dedicated exclusively to the study of geometry, disregarding Chapter IV about trigonometry. In the premodern copy, the textbook devotes chapters II and III to geometry teaching, totaling 186 pages, corresponding to 65.3% of the total pages of the volume. The modern textbook devotes chapters 3 and 4 and the entire Appendix to the study of geometry, corresponding to a total of 200 pages, or 63.7% of the total pages. In general terms, there was virtually no significant difference in the percentage of pages dedicated to teaching geometry in the two collections.

On the other hand, the overall analysis of the number of pages dedicated to geometry teaching indicates that, to comply with the 1951 Ordinance, even without including geometry teaching in the 1st and 2nd grades, for the 3rd grade, geometry content occupies more than 50% of the volume. In other words, in terms of the number of concepts to be addressed, they are concentrated in the 3rd and 4th grades and cover the main concepts of plane Euclidean geometry, developed in a deductive manner. Thus, an important first consideration to be made is that, unlike the current approach of diluting the geometric contents throughout all school years, in the 1950s and 1960s, geometry was concentrated in the last two middle school grades. However, in no way was it neglected or unexpressive in the textbooks analyzed, which also does not indicate an abandonment of the geometric field to the detriment of the fields of arithmetic and algebra.

It is precisely at the time of comparing the themes proposed in each collection that the differences become explicit. The contents distributed in Chapter II of the premodern textbook correspond, in the modern textbook, to Chapters 3 and 4. Chapter III in the premodern textbook, for the study of proportional lines and similarity of polygons, was transferred to Volume 4 in the modern collection. The concepts appearing in the premodern textbook in Chapter II are distributed in 150 pages, which correspond to Chapters 3 and 4 of the modern textbook, arranged in 186 pages, that is, 36 pages more dedicated to the same contents, as shown in Table 2 below.

Chart 2: Correspondence of geometry chapters in the two textbooks

Premodern textbook (Sangiorgi, 1958, p. 83-233)	**Modern textbook** (Sangiorgi, 1967, p. 111-297)
Chapter II – *Flat geometric figures. Straight and circle* 1. Geometric entities, propositions, congruence 2. Angles 3. Polygonal line, polygons 4. Triangles 5. Perpendicular and oblique 6. Parallel Theory 7. Sum of angles of triangles and polygons 8. Quadrilaterals, translation Circumference and circle 10. Arcs and angles. Geometric constructions	**Chapter 3** – *Study of geometric figures* Part 1 – Objectives /Geometric figures/ Topology Part 2 – Relationships and operations with point sets/Order structure/ Semi-Line, line segment, semi-flat /Segment measure Part 3 – Angle /Angle measurement/Complementary and supplementary angles Part 4 – Demonstrative practices/ Angles formed by two coplanar and one transverse lines **Chapter 4** – *Studies of polygons and circumference* Part 1 – Polygon / Diagonals / Triangles / Triangle congruence Part 2 – Logical construction of geometry / Proofs / Postulates / Theorems / How to make a proof / Reciprocal theorem/ Some theorems Part 3 – Quadrilaterals / Parallelograms / Trapezes / Fundamental theorems Part 4 – Circumference / Theorems / Circles

Source: Prepared by the authors from Sangiorgi (1958, 1967)

It is possible to observe that the sequence of contents discriminated in the column of the premodern textbook was preserved in the modern textbook with insertions of new topics. The total number of themes developed on geometry was reduced while the number of pages increased significantly. The same concepts started to be proposed with about 28% more space. Also noteworthy are the changes in the materiality of the modern collection, as Valente (2008) pointed out:

> New layout in the presentation of school content, in the use of fonts and numbers of varied sizes and shapes; inclusion of colors on internal pages, photographs, drawings. The aesthetics of the 1950s math textbooks are gone. The new collection, among other elements, also adopted color as information. (p. 30)

In any case, we identify that the same concepts gain another guise, both regarding the geometric knowledge in question, which we can designate as the *geometry to teach*, as well as regarding the methodological aspects introduced that could be designated by *geometry for teaching*. In our perspective, the two knowledges are intertwined; one justifies the other and needs to be examined side by side.

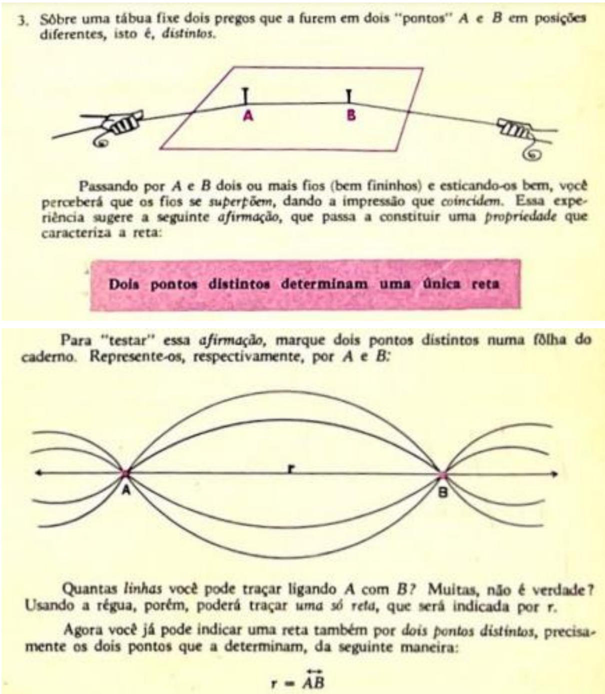
As already said, the 1950 and 1960 programs indicate a deductive geometry from the seventh grade of elementary school. In this way, the two collections portray proposals for teaching a kind of geometry in which postulates, theorems, and proofs are certainly present. Thus, we ask: What changes in such themes in the modern approach? As the number of pages increases, could we conjecture that the modern collection presents more postulates and theorems?

4 Examination of postulates

Regarding the postulates, both textbooks used this term as a synonym for axioms. In the premodern textbook, the term was introduced at the beginning of chapter II (in the topic "Group of geometric propositions: postulates and theorems"), on page 85, while in the modern textbook, the term first appears on page 234, in the second part of Chapter 4 (in the topic "What is postulated? What is a theorem?"). Thus, we observed that the list of postulates in the modern textbook was only presented after the geometry study proposed in Chapter 3, which deals with first notions and aims to prepare the student for the deductive approach.

By way of example, we can cite the postulate regarding the determination of a line that, in the premodern textbook, has its "direct" appearance, stated as: "Through two points passes a line and only one" (1958, p. 87), and the footnote complements "This postulate can also be enunciated as follows: two points determine a single line to which they belong." (1958, p. 87); and in the modern textbook, it appears first as "Exploratory exercise", in Chapter 3 (1967, p. 133), to be enunciated more than a hundred pages later as a postulate, on page 235.

Figure 1: Exploratory exercise on determining a line by two distinct points


3. Sôbre uma tábua fixe dois pregos que a furem em dois "pontos" *A* e *B* em posições diferentes, isto é, *distintos.*

Passando por *A* e *B* dois ou mais fios (bem fininhos) e esticando-os bem, você perceberá que os fios se *superpõem*, dando a impressão que *coincidem*. Essa experiência sugere a seguinte *afirmação*, que passa a constituir uma *propriedade* que caracteriza a reta:

Dois pontos distintos determinam uma única reta

Para "testar" essa *afirmação*, marque dois pontos distintos numa fôlha do caderno. Represente-os, respectivamente, por *A* e *B*:

Quantas *linhas* você pode traçar ligando *A* com *B*? Muitas, não é verdade? Usando a régua, porém, poderá traçar *uma só reta*, que será indicada por *r*.

Agora você já pode indicar uma reta também por *dois pontos distintos*, precisamente os dois pontos que a determinam, da seguinte maneira:

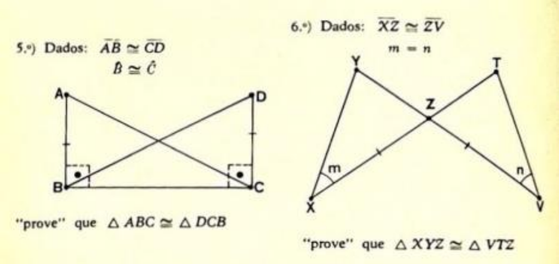
$$r = \overleftrightarrow{AB}$$

Source: Sangiorgi (1967, p. 133)

In the modern textbook, the same can be said about the cases of congruence of triangles. After introducing the concept of congruence indicating the six correspondences between sides and angles of two triangles, the author proposes exploring, through geometric constructions, situations in which the correspondences between only three elements of the triangles (always including one side) are sufficient to guarantee congruence. According to Sangiorgi, "the exercises concerning this 'economy' of corresponding elements, to know if two triangles are congruent, suggest the so-called Classical Cases of Triangle Congruence" (1967, p. 219). This occurs through a section of *Exploratory Exercises – Group 77*, with almost 20 pages (p. 219-227), ending with a *friendly reminder* in the form of a chart, which brings the statements of the four cases of congruence of triangles, with their respective acronyms.

It is worth mentioning that in the *Attention Test - Group 78* section that follows the exploratory exercises (p. 228), Exercise 3 proposes to justify the "why" of the congruence of the triangles that are part of the same figure and, after presenting a model example, all eight items that follow have in the statement the term 'prove' in quotation marks (cf. Figure 2).

Figure 2: Exercises to justify the congruence of two triangles



Source: Sangiorgi (1967, p. 230)

We interpret that the proposition of such exercises demonstrates a specific care of OS with the use of the term, since the justifications will involve congruence properties that have not yet been postulated nor proved but only inferred or verified experimentally through geometric constructions with ruler, compass, and protractor. Subsequently, in Chapter 4 (p. 235-236), when cases of congruence become part of the list of postulates, the term 'demonstration' begins to be used.

The choice for this approach was explicitly justified in the *Guide for teachers' use* [13] that accompanies Volume 3. According to Sangiorgi (1966),

> among the didactic issues that have caused a stir in the middle school course, there is the one that concerns the cases of congruence of triangles, formerly called classical cases of equality of triangles.
> *Is it postulated or proved?*
> In fact, neither one nor the other. (p. 50)

The excerpt above shows how attentive OS was, at the same time, to curricular issues (initiation to deductive geometry in the seventh grade of elementary school) and didactic-pedagogical aspects, explaining that in his textbook,

> the classic cases of congruence of figures *will be due to the "exploration" that students will do* in class, using ruler, compass, and protractor. Subsequently, in the axiomatic construction that will be made of geometry – *which will take place progressively* – these cases will be included in the postulate P10. (Sangiorgi, 1966, p. 50, emphasis added)

The premodern textbook, on the other hand, adopted a quite different approach to treat the congruence of triangles — close to that brought by Euclid in book I of *The Elements* — in which all cases of congruence are enunciated as theorems followed by their respective formal proofs. Referring to the 1st case of congruence, Sangiorgi warns us that "We could enunciate this case of equality and others as a *postulate*. Considering the age of the students, we prefer to introduce it as a *theorem*, admitting, in the meantime, the *postulate of the movement*" (Sangiorgi ,1958, p. 118). In fact, it was based on this postulate that the proofs of all cases were made and started by: "*Let us transport the triangle ABC over the triangle A'B'C' so that"* there is a

[13] We analyzed the 6th Edition, published in 1966.

coincidence between the congruent elements taken as a hypothesis (Sangiorgi,1958, p. 119).

It should also be noted that OS chose to introduce the concept of figure congruence at the beginning of Chapter II (p. 90) using the Euclidean principle of superposition[14], giving a more dynamic character (in the sense of movement) and biunivocal punctual correspondence to this concept. In Figure 3, follows the definition presented that confirms our interpretation.

Figure 3: Definition of congruence

CONGRUÊNCIA (*)

10. Figuras geométricas iguais ou congruentes. Duas figuras geométricas são *iguais* ou *congruentes* quando, por intermédio de um movimento, elas podem *coincidir*, isto é, se *superpor* (**). As duas figuras se superpondo fazem com que a cada ponto de uma delas *corresponda* um ponto da outra, e, portanto, a *todos os pontos de uma* correspondem *todos os pontos da outra*. Os pontos que se correspondem em figuras iguais dizem-se *correspondentes* ou *homólogos*.

Source: Sangiorgi (1958, p. 90)

In a note, the author warns: "**) The possibility of coincidence is guaranteed by *the Postulate of the Movement*" (Sangiorgi, 1958, p. 90, emphasis added by the author), introduced two pages earlier. We consider that the use of the *Postulate of the Movement*[15], at the beginning of the chapter, aims to give a less rigorous character to the definitions and propositions that follow, appealing to the intuition and empirical experience (evoked) of overlapping two figures by rigid displacement on the plane.

From this introduction, the comparisons of segments, angles, and triangles, among other geometric objects, are made similarly — supported by the *Postulate of the Movement* — and constitute the bases for the deductive development of the geometric properties that arise in the premodern volume.

In addition, the main differences between the premodern and modern textbooks, to institute a set of axioms aiming at initiating students to deductive geometry, were in the introduction of Archimedes' postulate and the postulate of division in the premodern textbook. In contrast, in the modern textbook, the definitions of segment and angle measurements occur in Chapter 3 (pages 146 and 159, respectively), preceding the axiomatic construction of geometry. A summary chart of the postulates presented in each textbook can elucidate what we found[16].

Chart 3: Set of Postulates enunciated in each textbook

Premodern textbook (Sangiorgi, 1958)	**Modern textbook** (Sangiorgi, 1967)
1st. There are infinite points, infinite straight lines, and infinite planes. On a straight line, there are infinite points; on a plane, there are infinite straight lines and, therefore, infinite points.	P1: TWO DISTINCT POINTS DETERMINE ONE AND ONLY ONE STRAIGHT LINE.

[14] Sangiorgi employs the reciprocal of the principle: "And the things that adjst on top of each other are equal among them" (Euclide, 1990, p. 178, our translation).

[15] "7th. A geometric figure can move in the plane (or in space) without deforming. (Postulate of the movement)" (Sangiorgi, 1958, p. 94).

[16] The postulates are listed in the order and format in which they appear in the textbooks. Common postulates (less in order) appear prominently in each column.

2nd. Infinite straight lines pass through a point.	P2: ON A STRAIGHT LINE THERE ARE AT LEAST TWO POINTS. THERE ARE AT LEAST THREE POINTS NOT ON THE SAME STRAIGHT LINE (NON-COLINEARES).
3rd. Through two points passes a straight line and a single one.	P3: THREE NON-COLLINEAR POINTS DETERMINE ONE AND ONLY ONE PLANE.
4th. Infinite planes pass through a straight line.	P4: IF TWO DISTINCT POINTS ON A STRAIGHT LINE BELONG TO A PLANE, THEN ALL POINTS ON THE LINE BELONG TO THE PLANE.
5th. Three points, not belonging to the same straight line, determine one plane and only one.	P5: IF B IS BETWEEN A AND C, THEN A, B AND C ARE COLLINEAR AND B IS ALSO BETWEEN C AND A.
6th. The line passing through any two points of a plane belongs to that plane.	P6: FOR TWO POINTS A AND C THERE IS AT LEAST ONE POINT B ON THE LINE AC, SUCH THAT C IS BETWEEN A AND B.
7th. A geometric figure can move in the plane (or space) without deforming. (Postulate of the Movement).	P7: THE LINE PARALLEL TO THE LINE r, DRAWN BY A POINT P THAT DOES NOT BELONG TO r. (famous postulate of Euclid!)
8th. A geometric figure is equal to the sum of its parts and greater than any of these parts.	P8: If X is between A and B, then: $m(\underline{AX}) + m(\underline{XB}) = m(\underline{AB})$.
Archimedes' postulate: "Given two unequal segments, there is a multiple of the smallest that outweighs the largest."	P9: IF P IS A POINT INSIDE THE ANGLE AÔB, THEN: m(AÔP) + m(PÔB) = m(AÔB).
Postulate: *Every angle has a bisector and a single one.*	P10: IF WITH TWO TRIANGLES OCCURS ONE OF THE CASES: L.A.L., A.L.A., L.L.L., L.A.Ao, THEN THE TWO TRIANGLES ARE CONGRUENT.
2nd. ***Division postulate***: *A segment or an angle can always be divided into any number of equal parts*.	Every line that passes through a point inside a circumference intercepts it at two points.
62. **Euclid's postulate** (or parallels 'postulate). *By a point outside a line as possible trace one, and only one, parallel to this line.*	

Source: Prepared by the authors based on Sangiorgi (1958, 1967)

Chart 3 explains the numerous changes made by OS in the two textbooks, particularly in the complete reorganization of the postulates. A significant difference concerns the concept of measurement, which in the premodern textbook is not mentioned in the postulates, while in the modern textbook, P8 and P9 mobilize this concept introduced in Chapter 3; that is, the concept of measurement is also another element in preparation for the approach to the deductive process, the object of study in Chapter 4.

The analysis of the postulates also reveals how much the author of textbooks, in our case, Sangiorgi, elaborates and adapts, following his objectives and conceptions, the way to produce a *geometry of teaching*. Despite being faithful to the principles of Euclid's book, the *geometry of teaching* presents itself as distinct and proper and constantly undergoes reformulations, evidencing the dynamic character of a school production.

In short, in the premodern textbook, the axiomatic-deductive approach to geometry is initiated from the first page of Chapter II, entitled "Plain geometric figures. Lines and circle" (p. 83); and in the modern textbook, it is introduced in Chapter 4 (second chapter dedicated to geometry), after a quite exploratory, experimental, and practical work, aiming to prepare students for such an approach, as we will see below. In our reading, Sangiorgi seems to alert to the importance of an intuitive geometry (or a geometric initiation) preceding the axiomatic construction, as recommended in the 1931 Campos and 1942 Capanema reforms, highlighted before.

5 Examination of theorems

Starting again with a quantitative analysis regarding the number of theorems proved in the two collections, contrary to what we conjectured by the increase of pages for the same topics, the study indicated a surprising drop. In our counting, we consider only the designation of "theorem" used in the two collections that have been proved in the theoretical and expository part since, in addition to these, other items have the designation "property", "reciprocal theorem", "corollary" or even "consequence", many of them equally proved. Well, in the premodern textbook, the number of theorems proved was 54 in Chapter II and, in the modern textbook, there is no proof of theorems in Chapter 3 and, in Chapter 4, the total of theorems proved goes to 28, that is, a practically 50% reduction in the number of theorems proved in the comparison between the premodern and modern textbook. What was OS's purpose in drastically reducing the number of theorems in a so-called deductive teaching proposal? And yet, what are the reasons for the formal proofs to start only in Chapter 4?

The author himself justifies the choice of the absence of proofs in Chapter 3 (according to him, to enable an experience for the student) and the change to another geometry, recommended by Hilbert in the section called *Pedagogical Observations*, referring to Chapter 4 of the *Guide for teachers' use of*:

> At this stage, the student is already "used" to verifying that some properties are consequences of other more elementary ones. It is the preparation for the beginning of a simple axiomatization, where the situations encountered in the exploratory exercises will be gathered in the form of postulates (axioms).
>
> Thus, a theory (advocated by Hilbert) is constructed from primitive concepts, postulates, and theorems, easily proven through a logical chain of reasoning.
>
> Therefore, the current treatment used to start deductive geometry is quite different from what was traditionally done. The student should no longer receive that "impact" in the first geometry classes of the seventh grade of elementary school: "What is a postulate?", "What is a theorem?" [...] mainly caused by the exclusive lack of maturity of the young student for such subjects.
>
> Only after much experience (it is enough to see that, in the textbook, the group of *ten fundamental postulates* is introduced after 120 pages dedicated to geometry) can the student be initiated into the *proofs*. (Sangiorgi, 1966, p. 51)

Sangiorgi clarifies that exploratory exercises were introduced in Chapter 3 (before starting the proofs) for the student to experience the verification of some properties as a result of others, that is, once again, we identified the attempt to insert an experimental geometry preparatory to the deductive study to be developed in Chapter 4. OS also indicates the construction of a new axiomatization for the formal proofs (recommended by Hilbert) in which the exploratory exercises carried out by the students in Chapter 3 become postulated at the time of demonstrating the theorems in Chapter 4. Everything indicates that the understanding of a deductive process is at stake, which incorporates "new postulates" to allow a more adequate understanding of logical-deductive reasoning for the students of the seventh grade. The object to be taught, in this case, deductive geometry, was modified both in its teaching object — *geometry to teach* — and also in the teacher's work tools — *geometry for teaching* — at the time of inserting exploratory exercises, configuring a *geometry of teaching*, elaborated by OS.

To illustrate the changes between the premodern and modern approaches, we present the theorem of the base angles of an isosceles triangle, as proved in the two collections. In the premodern textbook, it is the first theorem proved after the cases of congruence of triangles, which are considered theorems, and all are proved exactly 36 pages after the beginning of the study of geometry. In Chart 4, we reproduce his statement and the formal proof presented.

Chart 4: Theorem formal proof

38. Properties of the isosceles triangle.
a) Theorem: *In every isosceles triangle the base angles are equal.*
Considering the triangle *ABC* (fig. 72). We have:
H {*AB* = *AC*
T { $\hat{B} = \hat{C}$

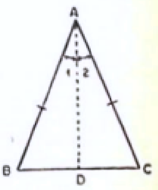


FIG. 72

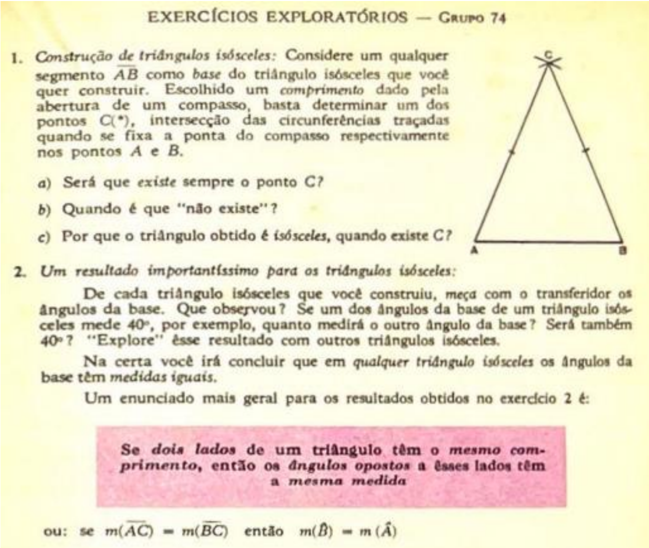
PROOF:
1. Let us trace the angle bisector of vertex A that meets *BC* at point *D*. Therefore: $\hat{1} = \hat{2}$ (bisector def.).
2. The triangles *ABD* and *ADC* are equal, by the 1st case of congruence (L.A.L.), and therefore are necessarily equal to the corresponding angles $\hat{B}$ and $\hat{C}$. Then:
$\hat{B} = \hat{C}$ c.q.d.

Source: Sangiorgi (1958, p. 119-120, emphasis added by the author)

In the modern textbook, this theorem is the first to be proved, in Chapter 4, 124 pages after beginning the study of geometry. The result of the thesis to be proved was studied and explored in Chapter 3 as an exploratory exercise (Figure 4a, p. 207); then in the attention test exercise (Figure 4b, p. 218).

The formal proof of the theorem is effectively done only in Chapter 4, as already said. However, because it is the first theorem and the first proof presented to students, it is developed in three pages (239 to 241), according to Figures 6 and 7 (a and b). Note that this theorem appears as a "model example" of topic 11 (p. 239) entitled *How to "face" a theorem successfully*, in which Sangiorgi discusses some aspects related to how to conduct a mathematical proof. We understand, therefore, that this text, destined for students, lends itself to announcing that they will be invited to produce their own proofs.

Figure 4: Exploratory exercises (a) and Attention test (b)



EXERCÍCIOS EXPLORATÓRIOS — GRUPO 74

1. *Construção de triângulos isósceles:* Considere um qualquer segmento $\overline{AB}$ como *base* do triângulo isósceles que você quer construir. Escolhido um *comprimento* dado pela abertura de um compasso, basta determinar um dos pontos C(*), intersecção das circunferências traçadas quando se fixa a ponta do compasso respectivamente nos pontos A e B.

   a) Será que *existe* sempre o ponto C?

   b) Quando é que "não existe"?

   c) Por que o triângulo obtido é *isósceles*, quando existe C?

2. *Um resultado importantíssimo para os triângulos isósceles:*

   De cada triângulo isósceles que você construiu, *meça* com o transferidor os ângulos da base. Que observou? Se um dos ângulos da base de um triângulo isósceles mede 40°, por exemplo, quanto medirá o outro ângulo da base? Será também 40°? "Explore" êsse resultado com outros triângulos isósceles.

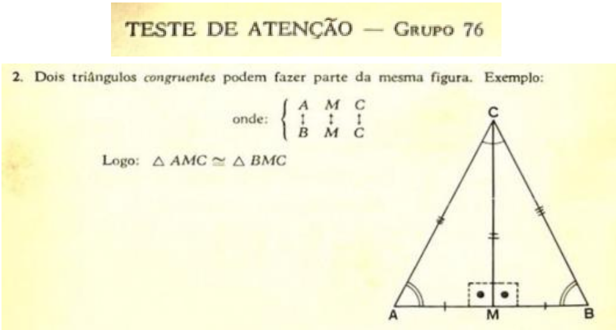
   Na certa você irá concluir que em *qualquer triângulo isósceles* os ângulos da base têm *medidas iguais*.

   Um enunciado mais geral para os resultados obtidos no exercício 2 é:

Se *dois lados* de um triângulo têm o *mesmo comprimento*, então os *ângulos opostos* a êsses lados têm a *mesma medida*

ou: se $m(\overline{AC}) = m(\overline{BC})$ então $m(\hat{B}) = m(\hat{A})$

(a)



TESTE DE ATENÇÃO — GRUPO 76

2. Dois triângulos *congruentes* podem fazer parte da mesma figura. Exemplo:

onde: $\begin{cases} A & M & C \\ \updownarrow & \updownarrow & \updownarrow \\ B & M & C \end{cases}$

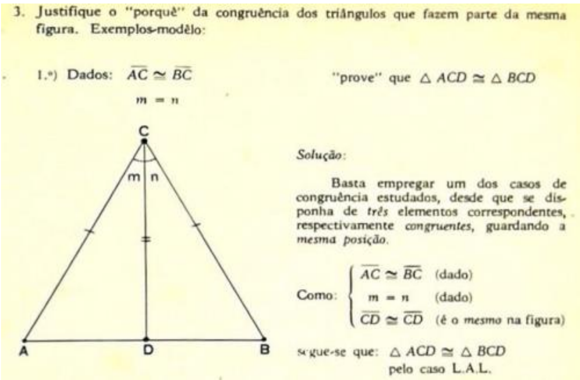
Logo: $\triangle AMC \simeq \triangle BMC$

(b)

Source: Sangiorgi (1967, p. 207 and 218)

In addition to the proposals identified above, Sangiorgi also proposes, later, another attention test, as shown in Figure 5.

Figure 5: Attention test



3. Justifique o "porquê" da congruência dos triângulos que fazem parte da mesma figura. Exemplos-modêlo:

1.º) Dados: $\overline{AC} \simeq \overline{BC}$ "prove" que $\triangle ACD \simeq \triangle BCD$

$m = n$

Solução:

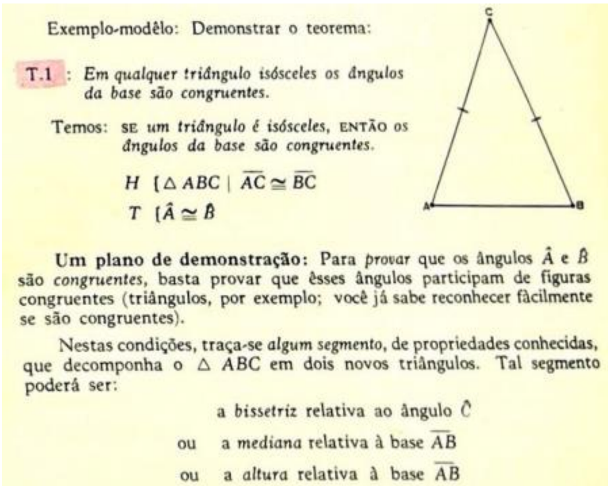
Basta empregar um dos casos de congruência estudados, desde que se disponha de *três* elementos correspondentes, respectivamente *congruentes*, guardando a *mesma posição*.

Como: $\begin{cases} \overline{AC} \simeq \overline{BC} & \text{(dado)} \\ m = n & \text{(dado)} \\ \overline{CD} \simeq \overline{CD} & \text{(é o mesmo na figura)} \end{cases}$

segue-se que: $\triangle ACD \simeq \triangle BCD$ pelo caso L.A.L.

Source: Sangiorgi (1967, p. 229)

Figure 6: Theorem T1 of the modern textbook



Exemplo-modêlo: Demonstrar o teorema:

T.1 : *Em qualquer triângulo isósceles os ângulos da base são congruentes.*

Temos: SE *um triângulo é isósceles*, ENTÃO *os ângulos da base são congruentes*.

H { $\triangle ABC \mid \overline{AC} \cong \overline{BC}$

T { $\hat{A} \cong \hat{B}$

Um plano de demonstração: Para *provar* que os ângulos $\hat{A}$ e $\hat{B}$ são *congruentes*, basta provar que êsses ângulos participam de figuras congruentes (triângulos, por exemplo; você já sabe reconhecer fàcilmente se são congruentes).

Nestas condições, traça-se *algum segmento*, de propriedades conhecidas, que decomponha o $\triangle$ ABC em dois novos triângulos. Tal segmento poderá ser:

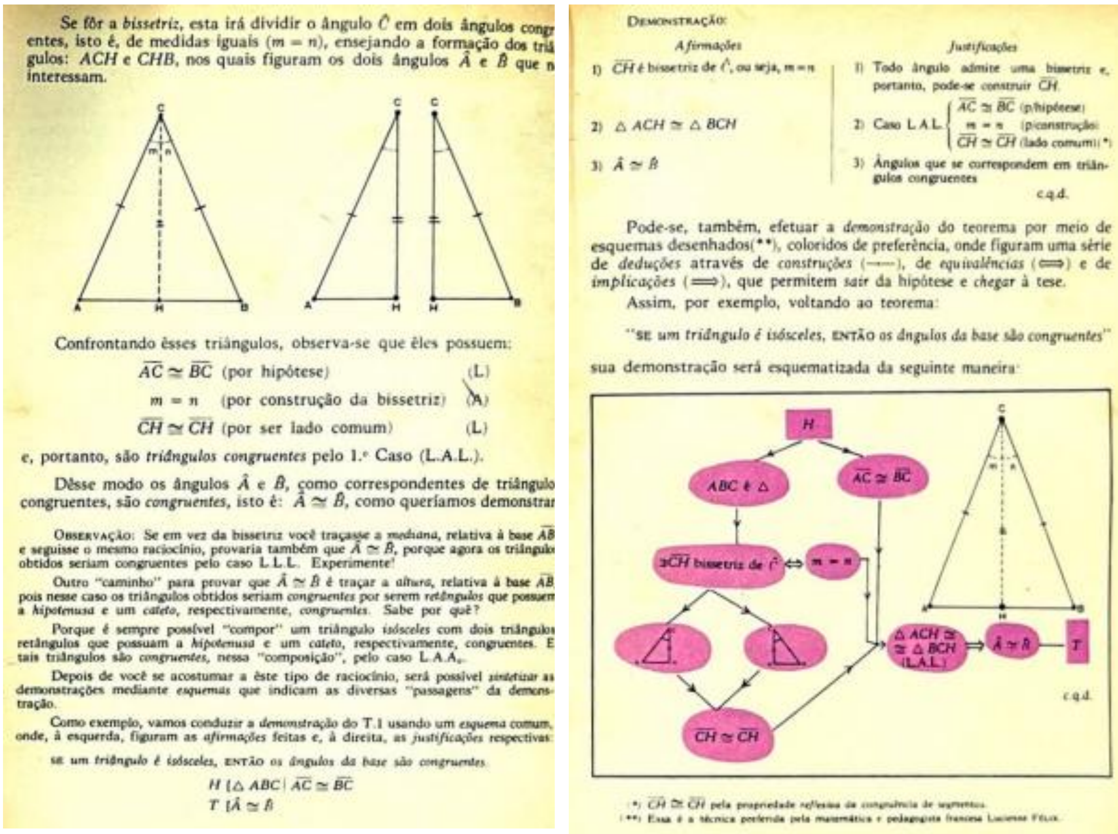
a *bissetriz* relativa ao ângulo $\hat{C}$

ou a *mediana* relativa à base $\overline{AB}$

ou a *altura* relativa à base $\overline{AB}$

Source: Sangiorgi (1967, p. 239)

Figure 7: Proof of the T1 Theorem



Se fôr a *bissetriz*, esta irá dividir o ângulo $\hat{C}$ em dois ângulos congr-
entes, isto é, de medidas iguais ($m = n$), ensejando a formação dos triâ-
gulos: *ACH* e *CHB*, nos quais figuram os dois ângulos $\hat{A}$ e $\hat{B}$ que n-
interessam.

Confrontando êsses triângulos, observa-se que êles possuem:

$\overline{AC} \simeq \overline{BC}$ (por hipótese) (L)

$m = n$ (por construção da bissetriz) (A)

$\overline{CH} \simeq \overline{CH}$ (por ser lado comum) (L)

e, portanto, são *triângulos congruentes* pelo 1.º Caso (L.A.L.).

Dêsse modo os ângulos $\hat{A}$ e $\hat{B}$, como correspondentes de triângulo
congruentes, são *congruentes*, isto é: $\hat{A} \simeq \hat{B}$, como queríamos demonstrar

Observação: Se em vez da bissetriz você traçasse a *mediana*, relativa à base $\overline{AB}$ e seguisse o mesmo raciocínio, provaria também que $\hat{A} \simeq \hat{B}$, porque agora os triângulo obtidos seriam congruentes pelo caso L.L.L. Experimente!

Outro "caminho" para provar que $\hat{A} \simeq \hat{B}$ é traçar a *altura*, relativa à base $\overline{AB}$, pois nesse caso os triângulos obtidos seriam *congruentes* por serem *retângulos* que possuem a *hipotenusa* e um *cateto*, respectivamente, *congruentes*. Sabe por quê?

Porque é sempre possível "compor" um triângulo *isósceles* com dois triângulos retângulos que possuam a *hipotenusa* e um *cateto*, respectivamente, congruentes. E tais triângulos são *congruentes*, nessa "composição", pelo caso L.A.A$_o$.

Depois de você se acostumar a êste tipo de raciocínio, será possível *sintetizar* as demonstrações mediante *esquemas* que indicam as diversas "passagens" da demonstração.

Como exemplo, vamos conduzir a *demonstração* do T.1 usando um *esquema comum*, onde, à esquerda, figuram as *afirmações* feitas e, à direita, as *justificações* respectivas:

SE *um triângulo é isósceles*, ENTÃO *os ângulos da base são congruentes*

$H \{\triangle ABC \mid \overline{AC} \simeq \overline{BC}$

$T \{\hat{A} \simeq \hat{B}$

(a)

Demonstração:

Afirmações	*Justificações*
1) $\overline{CH}$ é bissetriz de $\hat{C}$, ou seja, $m = n$	1) Todo ângulo admite uma bissetriz e, portanto, pode-se construir $\overline{CH}$
2) $\triangle ACH \simeq \triangle BCH$	2) Caso L.A.L. { $\overline{AC} \simeq \overline{BC}$ (p/hipótese); $m = n$ (p/construção); $\overline{CH} \simeq \overline{CH}$ (lado comum) (*)
3) $\hat{A} \simeq \hat{B}$	3) Ângulos que se correspondem em triângulos congruentes

c.q.d.

Pode-se, também, efetuar a *demonstração* do teorema por meio de esquemas desenhados(**), coloridos de preferência, onde figuram uma série de *deduções* através de *construções* (——), de *equivalências* ($\Leftrightarrow$) e de *implicações* ($\Longrightarrow$), que permitem *sair* da hipótese e *chegar* à tese.

Assim, por exemplo, voltando ao teorema:

"SE *um triângulo é isósceles*, ENTÃO *os ângulos da base são congruentes*"

sua demonstração será esquematizada da seguinte maneira:

(*) $\overline{CH} \simeq \overline{CH}$ pela propriedade reflexiva da congruência de segmentos.

(**) Essa é a técnica preferida pela matemática e pedagogista francesa Lucienne Félix.

(b)

Source: Sangiorgi (1967, p. 240-241)

The example elucidates the "novelties" incorporated in the modern textbook, compared to the premodern one. Since the exploration proposed in Chapter 3, in the exploratory exercise, in which students are invited to build different isosceles triangles, seeking a valid result in all of them, as well as in the attention test, in which students must seek to observe similar cases solved, constitute a preparation for the study of deductive geometry, even using the expression "prove", again, in quotation marks.

When he gets to Chapter 4, where the mathematical proofs is done, OS comments and explains each step. In addition, he uses new methodological strategies, which were not used before, such as the proof in two columns (statements and justifications) and the proof through drawn schemes, as shown in Figure 7b. It is necessary to highlight the appreciation of different resolutions and representations for proving the same theorem so that the student can understand the diversity of paths in the deductive process. A little later, still in Chapter 4, OS emphasizes the need for the student to produce his proof in a "friendly reminder":

> Don't "memorize" theorem proof!
> Value yourself, using any of the methods presented.
> Put your "touch" when implementing these methods, and you will be fulfilling in mathematics! (Sangiorgi, 1967, p. 258)

With the example presented and examined in detail, the need to reduce the theorems proved in the modern collection is explained since the proof treatment gains another perspective, much more explanatory, mobilizing different representation records for the logical chain. We can infer that such innovations correspond to how OS created, elaborated, from the moment he was immersed in the MMM debate, for a 13-year-old young person to understand the deductive process instead of memorizing the theorems and their proofs.

Another aspect to be considered in the modern approach concerns teachers, who started to rely on the *Guide for teachers' use*, as an important support to subsidize pedagogical practices, containing didactic guidelines and comments on students' frequent errors, among other contributions. In addition, the handbook provided the answers to the exercises proposed in the textbooks. In the case of the theorem proof, the handbook showed their development, which we can infer as pedagogical support for teachers to have more security in working with proofs in the classroom. In the guidebook, we explicitly find Sangiorgi's position that confirms our interpretations of the deductive approach and the justified decision to reduce the number of theorems: "Some *fundamental theorems*, responsible for the initiation of deductive geometry in the middle school, allow us to leave aside a huge series of "*theorems*" that, traditionally required of students, more represented a "dull" work — only "beat" them on the basis of "memorization" — than productive!" (Sangiorgi, 1966, p. 51, emphasis added by the author).

6 Final considerations

The article sought to examine and compare two didactic collections of the same author, recognized and of great editorial success, to bring more elements to understand the complexity of MMM, particularly concerning deductive geometry. The first conclusion that we can support is that the changes to the teaching of geometry presented in Volume 3 of each collection were significant and deserved to be highlighted in the modern approach, as Sangiorgi pointed out when claiming that the teaching of geometry is the "good bit" of the textbook. We understand that with the analyses produced, it was possible to show how OS elaborated alternative ways to develop proofs of theorems in geometry.

Osvaldo Sangiorgi combined two elements relevant for an author of textbooks to produce innovations: he had a solid mathematical background that allowed him the confidence to change postulates in joining a geometry supported by Hilbert, along with the summer course held in the USA, in which he most likely came into contact with debates and experiences between different geometries to be employed in teaching. On the other hand, he also had the expertise as a teacher of private lessons for students from renowned São Paulo schools, which allowed him to know the difficulties, students' most common errors, as he reports in the *Guide for teachers' use*. In other words, OS moved with skill both in *geometry to teach* and in *geometry for teaching*, which allowed him to announce pertinent recommendations to students and teachers about the *geometry of teaching*.

The second conclusion is related to the innovative character evidenced in our analyses: the modern collection of OS brought significant changes in the approach built for the deductive geometry proposed to middle school education, both in the scope of Euclidean geometry, of a *geometry to teach*, and in the methodological didactic aspects by proposing the insertion of exploratory and experimental exercises, in addition to the different records of representations as a way of preparing deductive geometry, which corresponds, in our view, to a *geometry for teaching*, which can be interpreted as an intuitive geometry, taken from the fifth and sixth grades of elementary school in the 1951 Minimum Program.

Jean Dieudonné (1906-1992), a French mathematician, leader of the Bourbaki group and who worked at USP, stated the following sentence that marked MMM: "If I wanted to summarize in one sentence the whole program I have in mind, I would do it with the slogan: Down with Euclid!" (O.E.C.E., 1961, p. 35). For us, Osvaldo Sangiorgi (1921-2017), a

Brazilian mathematician, leader of the Grupo de Estudos de Ensino de Matemática [Mathematics Teaching Study Group] (GEEM), graduated from USP, the phrase that would translate his appropriation of geometry of the middle school in MMM would be: “Do not ‘memorize’ theorem proofs!” (Sangiorgi, 1966, p. 158).

Although the analyzed collections have circulated throughout the state of São Paulo and beyond, given the considerable number of editions, we cannot, in any way, generalize or adopt the modern proposal of OS as that which represents or characterizes the modern teaching of geometry in Brazil. However, we cannot ignore the detailed analysis that the present article shows us: OS’s attempt to abolish the memorization of the proofs of theorems.

The complexity of MMM, associated with an analysis regarding public policies for teacher education, certainly needs to be inserted in the analytical context for a more reliable interpretation of geometry teaching in this period, as well as a historical representation of how geometry had been proposed in the reforms prior to the Movement.

Our analysis invites us to reflect that the debate on experimental-intuitive geometry versus axiomatic-deductive geometry has long permeated discussions on teaching geometry in middle school. Furthermore, consider that, in our analytical process of examining postulates and theorems, we are faced with the impossibility of an analytical process separate from the said *knowledge to teach* (*geometry to teach*) and *knowledge for teaching* (*geometry for teaching*) to produce a plausible story about the teaching proposals of deductive geometry from the perspective of Osvaldo Sangiorgi and to decipher his appropriations of the MMM.

Acknowledgements

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References

Bloch, M. (2001). *Apologia da História ou O ofício de historiador.* Rio de Janeiro, RJ: Jorge Zahar Editora.

Búrigo, E. Z. (2015). Professores de Matemática deveriam estudar história? In: *Anais do XII Encontro Gaúcho de Educação Matemática* (pp. 1-14). Porto Alegre, RS.

Caldatto, M. E. & Pavanello, R. M. (2015). Um panorama histórico do ensino de geometria no Brasil: de 1500 até os dias atuais. *Quadrante, 24*(1), 103-128.

Euclide (1990). *Les Éléments*, vol. 1, traduzidos do texto de Heiberg. Introdução geral por M. Caveing. Livros I-IV: Geometria Plana. Tradução de B. Vitrac. Paris: PUF.

Hofstetter, R. & Valente, W. R. (2017). (Org.). *Saberes em (trans)formação: tema central da formação de professores.* São Paulo, SP: Editora Livraria da Física.

Jahn, A. P. & Magalhães, G. R. (2023). As transformações geométricas em tempos pré-modernos: um estudo preliminar. In: *Anais do XV Encontro Paulista de Educação Matemática* (pp. 1-12). Guaratinguetá, SP.

Munakata, K. (2016). Livro didático como indício da cultura escolar. *História da Educação, 20*(50), 119-138.

Leme da Silva, M. C. (2008). A geometria escolar moderna de Osvaldo Sangiorgi. In: W. R. Valente (Org.). *Osvaldo Sangiorgi: um professor moderno*. (pp. 69-93). São Paulo, SP: Annablume Editora.

Leme da Silva, M. C. (2022). Abandono do ensino de geometria e a Matemática Moderna: uma revisão histórica. *Zetetiké*, 30, 1-22.

OECE (Organização Europeia para a Cooperação Económica). (1961). *Mathématiques Nouvelles*. Paris: OECE.

Pastor, E. S. & Leme da Silva, M. C. (2023). O ensino de geometria para o 1º ciclo do curso secundário em tempos pré-modernos. In: *Anais do XV Encontro Paulista de Educação Matemática* (pp. 1-13). Guaratinguetá, SP.

Pavanello, R. M. (1993). O abandono do ensino de geometria no Brasil: causas e consequências. *Zetetiké*, 1, 7-17.

Sangiorgi, O. (1958). *Matemática para a Terceira Série Ginasial.* São Paulo, SP: Companhia Editora Nacional.

Sangiorgi, O. (1967). *Matemática Curso Moderno.* 3º Volume para os ginásios. São Paulo, SP: Companhia Editora Nacional.

Sangiorgi, O. (1966). *GUIA para uso dos PROFESSORES*. 3º Volume – Matemática Curso Moderno. São Paulo, SP: Companhia Editora Nacional.

Santos, J. G. (2023). *Um estudo hermenêutico sobre as demonstrações no Movimento da Matemática Moderna*. 182f. Dissertação (Mestrado em Educação Matemática). Universidade Estadual Paulista. Rio Claro, SP.

Valente, W. R. (2008). Osvaldo Sangiorgi, um *best seller*. In: W. R. Valente (Org.). *Osvaldo Sangiorgi: um professor moderno*. (pp. 13-41). São Paulo, SP: Annablume Editora.

Valente, W. R. & Bertini, L. F. (2022). *A Matemática do ensino: por uma história do saber profissional, 1870-1960.* São Paulo, SP: Universidade Federal de São Paulo.