Teaching Maths using integrated metacognitive Instructions

Hans M. **Dietz**
University of Paderborn, Institute of Mathematics
Germany
dietz@upb.de

Summary

Large mathematics courses for students from applied disciplines are frequently affected by several obstacles, the most serious of them being many students' non-aptitude to read and understand the language of mathematics properly. At the same time, many students do not find appropriate working methods in time. In order to provide a remedy we introduced a system named "CAT" of in-teaching methodological instructions. Its key elements focus on building valid mathematical concepts. The paper presents some experience with CAT.

Key words: methodological instructions, mathematical language, metacognitive support, concept formation

Introduction

Since many years the basis courses "Mathematics for Economists" at the University of Paderborn count very large numbers of students. These students are faced with an unexpected – often as well undesired – huge amount of mathematics during their studies. At the same time, the level of mathematization both in the economical practice as in economical sciences has increased substantially over years. Thus, the mathematical requirements that economists have to meet have increased accordingly. In particular, students should be able to understand basic mathematical concepts as well as to master basic mathematical formalisms. In practice, however, many of them have great difficulties to cope with these requirements, although they perceive to invest great effort in their studies.

The author's long-run experience in teaching these courses indicates the main reasons for many students' difficulties: Inappropriate studying and working methods, the student's frequent non-aptitude to read and understand the language of mathematics properly, and, of course, a lack of previous school knowledge. In oder to overcome these difficulties we introduced a system of in- teaching methodological support. Its name “CAT” is derived from “check lists”, “Ampel” (German for traffic lights), and “toolbox” as basic elements. CAT has been introduced gradually since 2010 and accompanied by empirical studies assessing its performance. In the present paper we shall focus on how CAT supports reading and understanding the language of mathematics, being the main concern of CAT; other aspects of CAT go beyond the scope of this paper. For those we refer to Dietz (2013) and Dietz & Rohde (2012). Further details concerning the reading procedure can be found in Dietz (2012), (2013a) and Dietz & Rohde (2012a).

In Section 2 we shall give a brief overview of CAT, followed by a more thorough description of some of its main components and their practical use in Sections 3 to 6. In Section 7, we shall provide a survey of the experiences made so far, followed by conclusions in Section 8.

What is CAT?

CAT can be understood as the union of a teaching philosophy, certain instructional procedures and derived “products”. Our philosophy, first, considers teaching not mainly as a transfer of knowledge buth rather as a help for self-help in order to enable the students to acquire knowledge by themselves. Second, we do suppose only a quite limited amount of previous school knowledge and build up all what follows within the course. And third, we aim at a conscious knowledge management by the students. CAT’s basic procedures are three check lists, further the “Ampel” as a tool that supports self-control, and the “Toolbox” device as a support for problem solving. Two of the check lists are simply intended as reminders that recall all those working steps that should be performed weekly during the teaching terms as well as those that are recommended for the immediate preparation of exams; these shall not be dissussed within this paper. The third check list “reading” (CLR), however, is of substantial greater importance and subject of the following. The “products” to be mentioned are “vocabulary” and concept base. These have to be produced by the students themselves using the guidelines of the CLR, as we shall see in the sequel.

Vocabulary

The students are adviced to keep track of all notions and symbols that are introduced within the course by including them in a vocabulary. We recommend to use the following pattern:

Vocabulary:	
Key word:	$\subseteq$
Definition:	$A \subseteq B :\Leftrightarrow (x \in A \Rightarrow x \in B)$ with A, B sets
Description:	set inclusion symbol
Read:	“A is a subset of B”

Thus, the vocabulary can be seen as material lexicon. It is intended as a basic and reliable reference for all further occurences of the corresponding notion or symbol. It is clear that this function can be only accomplished if all students carefully augment their vovabulary.

The check list “reading” (CLR)

The CLR is to support reading mathematical expressions in such a way that a valid mathematical concept is obtained. Imagine, e.g., that a student works through his lecture notes and faces the following expression:

$H := \{ A \mid A \subseteq M \}$, (1)

where M denotes a previously given set. Of course, in a good series of lectures the concepts of “definition”, “set” and “set inclusion” as well as the symbols ⊆and {.|.} should have been introduced beforehand; thus there should be corresponding entries in the student’s vocabulary. Now the CLR foresees five steps as follows:

Translating stage:	Conceptual stage:
• $S1$ *read*	• $S3$ *animate*
• $S2$ *play*	• $S4$ *illustrate*
	• $S5$ *talk*.

The first stage brings (1) in a form that can be read out fluently. Frome there the second stage builds up a valid mental concept. The whole process starts by identifying (1) as a string of non-decomposable symbols:

H	$:=$	$\{$	A	$\mid$	A	$\subseteq$	M	$\}$
···	···	···	···	···	···	···	···	···

In the first step S1 we assign to each of these symbols its correct meaning. Note that for this purpose a well-defined reliable “database” has to be used consciously (when reading lecture notes, usually it is given by the full lecture context – including vocabulary – and reliable sources cited within it). For each symbol with a database entry its meaning is imported from there; if necessary, desambiguation is accomplished by cross-comparison between boxes. The remaining symbols do not possess a pre-defined meaning but either obtain a meaning by the expression itself or rather play a syntactical role3 that is identified and described by S1. The result of S1, although being correct, may never-theless look somewhat ugly. Therefore S2 brings this result in a form that can be read out fluently. In our example we might arrive at

H	$:=$	$\{$	A	$\mid$	A	$\subseteq$	M	$\}$
H (new)	is defined as	the set of all	A	with the prop-erty that	A	is a subset of	M	

An even shorter form of this sentence, obtained solely by means of our natural language, is “H is the set of all subsets of M”. Although this sentence may seem to be easily accessible, our experience indicates the contrary, i.e., many students still do not understand the underlying concept because of a too poor concept image in the sense of Tall & Vinner (1981). Therefore we foresaw the conceptual stage: S3 animate aims at providing examples and non-examples, and S4 at providing visualizations, if appropriate. In our lectures as well as in Dietz (2013a) we give detailed instructions how to obtain these. The S5 step should be performed by each student himself.

Concept bases

In order to support the students when performing the conceptual stage we offered a form that should be filled by the students themselves. An example of a filled form is displayed in Figure 1. The black parts represent the given form; the blue ones have been filled in. The upper part, shaded in light gray, is just the entry of he vocabulary. The middle part contains the extensions created by S3 andS4. Note that the last part goes beyond this; here connections to other notions and applications, to be worked out within the further course. Working with these forms is trained several times within the weekly exercise groups.

Concept base:

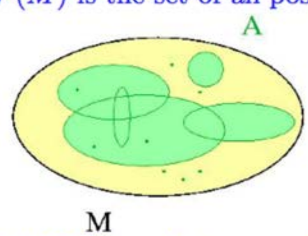
Vocabulary:	
key word:	$\mathcal{P}(M)$ (M a given set)
...	...
Extensions:	
Examples:	$\mathcal{P}(\emptyset) := \{\emptyset\}$, $\mathcal{P}(\{1\}) := \{\emptyset, \{1\}\}$, $\mathcal{P}(\{1,2\}) := \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, ...
Non-examples:	Attention: $\mathcal{P}(\emptyset) \neq \emptyset$
Visualisation:	$\mathcal{P}(M)$ is the set of *all* possible A like this: 
Important statements:	(a) *If M is a finite set with n elements then $\mathcal{P}(M)$ has 2^n elements* (b) $A \subseteq B \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$ ...
Applications:	(to follow later in the course)

Figure 1. Example of a concept base

CAT in practice

As mentioned, CAT became integrated part of the course “Mathematics for Economists”. This course is organized as follows: Plenary lectures for up to 1400 students, two times weekly, are accompanied by up to 40 exercise groups for 30-40 students each, taking place once in week. Besides that, every week students can optionally submit written solutions to certain tasks in order to have them corrected. Moreover, there is a course-specific mentoring system that can be used voluntarily by interested students.

At first, CAT’s basic elements are introduced within the lectures. In particular, how to create vocabulary entries and how to use the CLR is demonstrated frequently, including the creation of examples, non-examples, and visualizations. Also, CAT is subject to the exercise groups, where “reading maths”, creating concept bases, using the tool box etc. is exercised. Moreover, there are written exercises aiming at reading, concept bases and more. Last but not least much attention is spent to integrate the CAT concept in the mentoring processes, too. Here students should not only – and not primarily – receive answers to mathematical questions but rather to the question, how to become able to answer the mathematical question themselves.

Again it should be emphasized that the main focus of CAT is on the reading coprehension. This corresponds fully both to the author's own experience w.r.t. the lack of corresponding abilities and to observations from the literature, e.g. those of Österholm (2006). As a particular problem we address "silent conventions" of the language of mathematics, as described by Hefendehl-Hebeker (2013).

It is obvious that teaching the elements of CAT within the regular mathematical lectures deserves a certain amount of time that is no longer available for teaching proper mathematical content. The idea – and desire – when designing CAT, however, was that a better studying and understanding performance promoted by CAT would allow for re-gaining this loss of time. During the first courses with CAT it turned out that a good performance cannot be achieved without a thorough preparation of the whole teaching team, including those students that give tutorials or correct written exercises. Thus an additional training of the teaching team had to be established. Admittedly, the effort to present CAT in an adequate way was by no means neglectable. Moreover, the gradual introduction of CAT was accompanied by a several studies that investigated the acceptance and performance of it.

Some experience

The purpose of CAT was to help the students to study more efficiently and to understand and practice mathematics better and deeper. At the same time, the working methodology should provide a surplus value also for non-mathematical disciplines. On the other hand, it is obvious that CAT cannot be obtained for free; it requires time and effort both from the teaching personnel and the students. Hence it may be questioned, whether and to what extent CAT does improve the situation as well as what possibly hinders CAT from performing better. Last but not least one may ask whether the effort for practising CAT is adequate and justified. Up to know, we do not have a final answer to these questions, but they can be answered at least in part. Doing so, we refer to

- the immediate feedback from students and tutors
- representative samples from written exams of February 2012-2014
- insights from several studies.

In particular, the latter are

- a qualitative study „ECOStud" (St1), 2010 – 2012
- a preliminary quantitative survey (St2), January 2012
- a main quantitative survey (St3) in two parts:
  (St3 a) prospective part, January 2014
  (St3 b) retrospective part, June 2014.

ECOStud was a qualitative study following a complex design of combined methodological training and interviews, conducted together with J. Rohde, cf. Dietz & Rohde (2012). The surveys were questionnaire based, anonymous, and focussed on CAT's acceptance. Typically we obtained 500-800 answer sheets from students. The analysis of (St3 b) and of an additional survey of January 2015 is still in progress.

The purpose of the present paper is *not* to provide a detailed description of all of these studies but rather to point out some of the main insights that could be gained from all sources of information as a whole; further details from the studies can be found, e.g., in Feudel (2015). Our experience with CAT made so far allows to claim that

(I) *The objectives of CAT do meet the students' demand.*

This is particularly true with respect to CAT's reading procedure. Within the ECOStud project it could be seen that most of the participants were not able to cope with simple mathematical expressions without CAT's support. But also in the main survey of January 2014[1] we find 84,7% answers10 that tend to agree to "I approve that reading mathematical expressions is trained[2]." At the same time, only a small minority of students believe to have (successful) own techniques. Further we have 89.1% and 87.9% that tend to say that for them "knowing definitions" and "understanding mathematical notions exactly" is of importance; both being subject to the vocabulary and concept base.

(II) *CAT does help the students, although not yet to the desired extent.*

Let us look at the effects in "reading" first: As the samples from the examinations show, after we introduced the CLR technique the percentage of students that can correctly read out a given mathematical expression moved up from just under 50% up to about 70%. We conjecture that the last number is near to some natural maximum. The concept understanding, however, initially stayed far below at about 10% of the students. Not until the introduction of the concept base device in the winter term 2012/13 this percentage increased up to 20%. Interestingly, we saw a synchronous increase in the ability to reason correctly. In addition, some students reported to find concept bases helpful to such an extent that they began to use them also in other disciplines, e.g., in statistics. – Nevertheless, the percentage of students that reach full concept understanding is still too low. Moreover, there is not significant global improvement in the examinations' results. As the results of (St3a) suggest, still many students have problems to appropriately assess their own level of understanding of the lectures' topics. At the same time, the "Ampel" is not well-understood and thus employed only by a quite small part of the audience. As a consequence, the importance of concept bases and other instruments is frequently underestimated, too. Summing up, we see some progress, but still a promising rather than a fully satisfying one.

(III) *The power of CAT could be exploited better by improved communication.*

It turned out that the instruments of CAT and, in particular, the steps of the CLR have been understood and adopted to quite different extents. Concept base, toolbox and the check lists "reading" and "weakly working steps" as a whole are used more or less by about one third of the students, while the vocabulary is best accepted instrument (more than 70%); on the other hand, the Ampel is the least one (15.6%). The single steps of each instrument in turn are differently accepted as well. Altogether the degree of acceptance of CAT's instruments still cannot satisfy, although it has increased substantially – in fact, nearly doubled – in comparison to 2011/12. The reason for this increase was the integration of CAT into the tutorials, the main reason for the increase to be not even larger is that many students do not know or understand these instruments sufficiently. Thus, further progress should be possible by going further in the right direction.

To be more specific with respect to "reading mathematics", we underline that 63,8% of the students tend to agree to "Working with CLR should be demonstrated more frequently within the

[1] The survey was analyzed by Frank Feudel along the author's guidelines.
[2] German: "Ich finde es gut, dass das Lesen mathematischer Ausdrücke geübt wird."

course[3].” Apparently these refer mainly to the step S3 animate, i.e. example generation, that was not quite understood by 61.0%; even more (78%) find it difficult to generate own examples. As this step had been demonstrated several times in the lectures but was not subject to the weekly exercises, it seems natural that an explicit treatment in the exercises could promote the understanding of this technique.

The latter conjecture is supported by the sampled results of the February 2014 exam concerning a reading-to-concept task, with the following expression given:

$$M \ := \ \{ \ x \in \mathbb{R} \mid \forall y \in \mathbb{N} : x \leq y^2 \ \}.$$

The students were asked to (a) read out this expression, (b) provide examples of elements of the set *M*, (c) provide examples of numbers not within this set, and (d) describe this set as briefly as possible in own words or with another expression. For each of these subtasks the students might obtain 0, 0.5 or 1 points. Given that a student solved one of these subtasks at a given level (left column), we obtained the following empirical probabilities to solve one of the other subtasks in the right column[4]:

Table 1. *Conditional probabilities*

ability (in points) to (a) read out	probability of correct (b) example generation
1	$\approx$ 42%
0.5	$\approx$ 37%
0	$\approx$ 31%
ability (in points) to (b) generate examples	probability of correct (d) concept understanding
1	$\approx$ 48%
0.5	$\approx$16%
0	= 0%

Source: Written course exams, Feb. 2014

We see that the ability to achieve a full concept understanding relies heavily on the ability to generate examples, and the this ability in turn is supported by correct reading out. Thus the training of example generation deserves prior attention.

(IV*) CAT’s performance heavily depends on its full integration in all teaching levels.*

By “integration in all ... levels” we mean that CAT has to be addressed not only in the lectures, but as well in the exercise groups (tutorials), in the mentoring system and, as far as possible, also in the correction of written exercises. (St2) indicated a strong decline of CAT’s acceptance if CAT is not sufficiently present in the tutorials. Moreover, (St3a) exhibits a strong correlation between the acceptance and usesage of CAT and the tutoring person. Hence we need a “full” integration of CAT in all levels, i.e., all members of the teaching team should be fully behind it. This requires an extensive and careful coaching process.

(V) *The students’ workload caused by CAT is reasonable.*

From (St3a) it could be seen that many working steps connected with CAT can be – and are – performed within a very small amount of time, with the only exception of concept bases. On the average, students do need not more than 30 minutes to create a concept base. But also this time, usually to be spent about twice a week, is acceptable. Moreover, 48.5% students

[3] German: “Die Arbeit mit der CLL sollte in den LV häufiger demonstriert werden.”
[4] Here, we understand “correct” in the wider sense of achieving at least .5 points.

themselves see a benefit of concept bases as useful surveys.

(VI) *The total working time of many students is insufficient.*

(St3a) showed that just below 70 % of the students do not invest enough time in their self studies, even when being measured in terms of the 5 ETCS "value" of the the course. Note that this statement relies on informations that the students' themselves provided voluntarily. It may seem somehow surprising as many students perceive or at least claim their own effort to be quite hard. Here we see a clear discrepancy between factual and perceived reality. To the author's opinion, this effect can be explained in part by the cultural "shock" provoked by the transition from school to the university. Probably, the students do have incorrect expectations about the amount of effort that their studies really require. Moreover, excessive use of e-mail, mobile phones, etc. at any time causes time losses that may be substantial without being perceived so.

(VII*) Poor memorization becomes an increasing problem.*

This claim adresses the author's observation that knowledge does not persist "very long" in the students' memories. This refers both to factual and to methodological knowledge, and "very long" does often mean a very short period of time, say 2-3 weeks. The disadvantageous influence hereof is at hand, because the students' mental vocabularies, concept bases, and methodological skills do rapidly fade out. As a consequence, many students cannot benefit in the following terms from the knowledge and skills acquired within the first term. In the author's perception, the significance of this phenomenon has increased substantially over years. This perception is also supported by results from written exams as, e.g. the written final exam of the courses 2nd term (July 2014), where it could be seen that up to 90% of the participants did no longer have basic knowledge from the 1st term at their disposal. Although subsequent exams differ from each other, the order of magnitude of this information loss was the largest so far. It should be noted that our observation concerning increased memorization problems is supported by observations of increasing frequencies of basic errors, that were documented, e.g., in Weinhold (2013).

8. Conclusions

From the above we can conclude so far:

- Teaching CAT, in particular "reading mathematics", does meet a present demand.
- Presumably, a better performance can be achieved by better communicating some of its elements. In particular, S3 and S4 deserve much attention.
- It seems worthwile to further investigate the phenomenon of poor memorization and to develop strategies that cope with it.

Moreover, the teaching of mathematics should be connected with clear expectations w.r.t. the students' workload and own effort.

Acknowledgement

The author wishes to thank Janna Rohde and Frank Feudel for their engagement in accomplishing and analysing the accompanying studies.

References

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52, 215–241.

Cox, R. (1999): Representation construction, externalised cognition and individual differences.

Learning and Instruction, 9, 343–363.

Dietz, H.M. (2012). *Das ECOMath-Handbuch. Mathematik für Wirtschaftswissenschaftler* (in German). Springer, Heidelberg et.al.

Dietz, H.M. (2013). CAT – ein Modell für lehrintegrierte methodische Unterstützung von Studienanfän- gern (in German). Paper presented at the 2nd khdm Working Conference *"Mathematik im Übergang Schule/Hochschule und im ersten Studienjahr"*, Paderborn, February 23, 2013. To appear in Biehler, R. et al. (Eds.): „Lehren und Lernen von Mathematik in der Studieneingangsphase – Herausforderungen und Lösungsansätze", Springer, 2015.

Dietz, H.M. (2013a): "Semiotics in 'Reading Maths' ". In G. Kadunz (Ed.), *"Semiotische Perspektiven auf das Lernen von Mathematik"*, Springer Masterclass, Heidelberg, 2015, 187-205.

Dietz, H. M. and Rohde, J. (2012). Studienmethodische Unterstützung von Erstsemestern im Mathematikservice (in German). *Beiträge zum Mathematikunterricht,* 2012.

Dietz, H. M. and Rohde, J. (2012a). Adventures in Reading Maths. In: *Philosophy, Mathematics, Linguistics: Aspects of Interaction.* Proceedings of the 2012 PhML Conference, Steklov Institute of Mathematics and Euler Mathematical Institute, St. Petersburg, 61-68.

Feudel, F. (2015). "Studienmethodische Förderung in der Mathematik für Wirtschaftswissenschaftler - Chancen und Schwierigkeiten". *Beiträge zum Mathematikunterricht,* 2015.

Hefendehl-Hebeker, Lisa (2013): Mathematische Wissensbildung in Schule und Hochschule – Gemein- samkeiten und Unterschiede (in German). Paper presented at the 2nd khdm Working Conference *„Mathematik im Übergang Schule/Hochschule und im ersten Studienjahr"*, Paderborn, February 23, 2013. To appear in Biehler, R. et al. (Eds.): „Lehren und Lernen von Mathematik in der Studienein- gangsphase – Herausforderungen und Lösungsansätze", Springer, 2015.

Österholm, M. (2006). Characterizing Reading Comprehension of Mathematical Texts. *Educational Studies in Mathematics 63* (3), 325-346.

Tall, D. and Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12,* 151–169.

Weinhold,C. (2013). Schwierigkeiten von Lernenden beim Übergang ins Studium (in German). Paper presented at the 2nd khdm Working Conference *„Mathematik im Übergang Schule/Hochschule und im ersten Studienjahr"*, Paderborn, February 23, 2013. To appear in Biehler, R. et al. (Eds.): „Lehren und Lernen von Mathematik in der Studieneingangsphase – Herausforderungen und Lösungs- ansätze", Springer, 2015.