Conceptualizing Societal Aspects of Mathematics in Signal Analysis

Reinhard Hochmuth, Stephan Schreiber
Leibniz Universität Hannover, Leuphana Universität Lüneburg

This paper focuses on different possibilities for conceptualizing didactically relevant aspects of advanced mathematical subject matter in such a way that fits with subject scientific categories considering mathematically learning as a societal mediated process according to Holzkamp (1993). This concern requires studying subject matter, teaching and "university" not as conditions that cause reactions but as meanings in the sense of generalized, societal reified action possibilities. We consider the following three theoretical lenses: an epistemological-philosophical analysis, Anthropological theory of didactics and elements of Bernstein's theory. After rather short characterizations of these approaches we illustrate their application to the introduction of the delta distribution in a signal analysis text book.

Introduction

This paper contributes to an ongoing major research project that describes and analyzes form and content of advanced mathematics and its teaching and learning from a "subject scientific" point of view. This approach grounds in the so-called "Critical Psychology", worked out in Holzkamp (1985), also see Tolman (1991). Recently this theory becomes internationally more known within the mathematics education community due to Roth & Radford (2011), who value "German Critical Psychology" as a further development of the culture-historical approaches by Leontjev (1978) and Vygotsky (1978).

The main features of "Critical Psychology" and its subject scientific point of view are well elaborated psychological categories for describing and analysing subject related experiences, in particular thoughts, actions and learning, in such a way that major societal aspects are inherently incorporated. Within this framework there is so far not much (if any) research done that relates to mathematical learning in the context of higher education. This paper focuses on different opportunities for conceptualizing (specific aspects of) advanced mathematical subject matter in such a way that it is "viable" to descriptions and analyses based on subject scientific categories, i.e. that allows to consider mathematical learning in a specific way as a societal mediated process or activity. The main connecting link between the subject scientific approach and subject matter is a specific understanding and use of the concept "meaning". Consequently, a criterion for the adequacy of an approach for the analysis of subject matter is to blend with crucial aspects of the subject scientific concept of meaning. In view of this criterion we exemplarily discuss the application of analysing tools from anthropological theory of didactics (ATD), Bernstein's discourse analysis and materialistic oriented epistemological studies by means of the introduction of the delta distribution in signal analysis.

The structure of the paper is organized as follows: At first we sketch concepts from the subject scientific theory that are relevant for an embedding of our further observations; in particular we describe shortly their specific concept of meaning. Then we present a few details about an introduction of the delta distribution in signal analysis and demonstrate in the following, how an epistemological-philosophical approach, ATD and elements of Bernstein's theory could contribute to working out basic facets of corresponding meanings. A summary and remarks on further research steps conclude the paper.

Subject Scientific Approach ("Critical Psychology")

Critical Psychology claims to present a scientific discussable and criticizable elaboration of basic psychological concepts (categories). The starting point is a historical-empirical investigation of general historical-specific characteristics of relations between societal and individual reproduction. Within the context of this paper there are two important points to notice: First, the actual historical-specific form of subjectivity is characterized by the so called "possibility relation" with respect to the societal reality, which gives and includes in particular the basic experience of intentionality and makes consciousness to a prerequisite for the societal reproduction. Second and connected to the first, the specific modality of subjective action experiences comprises a certain discourse form ("I" speak about my "own" actions in terms of subjective reasonable actions and of premises in the light of "my" living interests.) that characterizes to some extent the specific subject scientific standpoint.

According to this modus, world conditions are given in terms of meanings in the sense of generalized societal action possibilities. Meanings of reality aspects, which are relevant for "me", become premises. Consequently, subject scientific considerations are essentially given by premises-reasons-relations.

In "Critical Psychology" meanings and their mediation role do not only represent social-interactive but also societal aspects grounded in relations between production and reproduction. Via meanings, human activities, like teaching and learning, can be thought as societal mediated. An analysis of subject activities regarding its societal mediation requires an adequate conceptualization of the objective situation of the subject. As representations of subjectively relevant objective conditions they have to be describable and analyzable as generalized action possibilities, hence meanings.

Since meanings appear (via objective-subjective premises) to some extent as the medium within which subjective action reasoning is grounded, their study is a prerequisite for describing and analyzing related cognitive, motivational and emotional processes as aspects of subjective activities like learning under concrete societal conditions.

Meanings in the indicated sense are relevant for acting and thinking, but do not determine them. Furthermore, they are not only of linguistic-symbolic nature, but objective-societal objects, to which symbols relate. Of course, in particular in mathematics and science, symbols are objects by their own and acting with them underlies rules that are epistemologically and institutionally determined and are also determined as elements of a scientific or pedagogical discourse.

In the following, we exemplarily consider the introduction of the delta distribution in signal analysis and apply three different approaches shedding light on different aspects of meanings in the indicated sense of generalized societal action alternatives.

Higher Mathematics and Signal Analysis: The Introduction of the Delta Distribution

Students in engineering courses learn mathematics in at least three contexts. First, they have to pass courses in higher mathematics. Here the students learn mathematical concepts from analysis, linear algebra and sometimes elementary numerical analysis. These topics are mostly presented in a more or less theoretical mathematical setting. Second, students must apply mathematics in their basic engineering courses. Since most of the mathematical concepts required in these courses have not been presented in the higher mathematical courses until that moment (at least in Germany), often additional seminars are offered that accompany the engineering courses. While exact mathematical definitions and/or justifications for the mathematical concepts used in the basic theory-oriented engineering courses are often presented later in the higher mathematics courses, this is generally not the case for more advanced mathematical concepts applied in the third context, e.g. courses like signal analysis. For example, a concept like delta-distribution is typically not covered in the mathematical courses attended by an electrical engineering student. It is not clear how, if at all, students are able to integrate these variations of mathematics. To study this problem, it would be helpful to have research-methods that represent, relate and reflect these variations of meaning.

Next we sketch exemplarily the introduction of the delta-distribution in one textbook, Frey and Bossert (2008): Continuous linear and time-invariant systems are generally described by differential equations. In view of discontinuous signals that are not differentiable in single points, the authors introduce so called "generalized functions" or "distributions" (p. 108). This leads to the problem introducing some kind of derivative to a non-differentiable function, which is exemplarily considered by means of the Heaviside function

$$\varepsilon(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases},$$

that is even not continuous at t=0. Supported by graphical illustrations it turns out that the Heaviside function can be represented as the pointwise limit of the parameterized sequence of differentiable functions

$$f_a(t) = \frac{1}{\pi}\left(\arctan\left(\frac{t}{a}\right) + \frac{\pi}{2}\right), \quad a > 0,$$

for a→0. These functions are formally differentiated and under the not further justified assumption that function limits and differentiations are commutable it is obtained

$$\frac{d}{dt}\varepsilon(t) = \frac{d}{dt}\lim_{a\to 0} f_a(t) = \lim_{a\to 0}\frac{d}{dt} f_a(t) = \lim_{a\to 0}\frac{1}{\pi}\frac{a}{a^2 + t^2}.$$

Here the convergence of the starting function sequence can be understood pointwise which fits to the convergence concept in higher mathematical courses for engineers. Instead, the limit of the differentiated sequence turns out to be a new object that mathematically has to be interpreted as functional on a space of test functions. Furthermore, the related limit has to be understood as distributional, which is distinctively different to the limit concepts introduced in higher mathematical courses for engineers. Considering the graph of the differentiated function for a→0 the notion

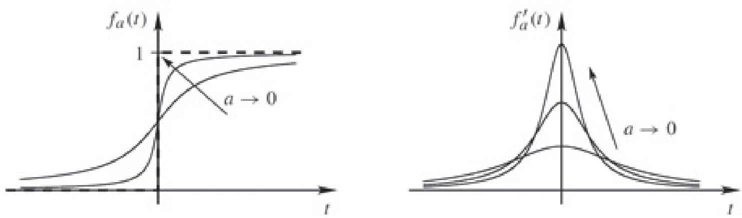

Figure 1: Representing the Heaviside Function and the Dirac Impulse as Limit (Frey & Bossert, 2008, s. 109)

"impulse" is suggested and for the "limit impulse" the symbol $\delta(t)$ with name "Dirac impulse" is introduced as the pointwise limit function

$$\delta(t) = \lim_{a\to 0}\frac{d}{dt} f_a(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}.$$

Additionally, the effect of the Dirac impulse is symbolically represented by the integral $\int_{-\infty}^{\infty} \delta(t)\cdot\phi(t)dt = \phi(0), \text{ for } \phi(t) \in C_0^\infty(\mathbb{R}^n)$, which relates to the concept image (Tall & Vinner, 1981) of an integral as infinite sum of infinitesimal small pieces weighted by the "function" $\delta(t)$.

From a strong mathematical point of view severe mathematical deficits in the foregoing argumentation are obvious. Taking into account the, in the 20th century, well-developed theory of distributions another and mathematical correct presentation could be given. E.g., integrals with respect to distributions might be introduced that have (nearly) nothing to do with the Riemann integral taught in higher mathematical courses that represents the reference concept here. In the context of this paper we refer to those deficits in another way: We argue that there are very specific reasons for those inconsistencies. E.g. in the following section, we refer to epistemological-philosophical ideas claiming that they root (in a certain sense) in necessities that are related to the application of mathematics to reality, i.e., the use of mathematics in empirical sciences as a mean for developing knowledge facilitating measurements and calculations.

Epistemological-Philosophical Analysis

Epistemological-philosophical analyses contribute to a deeper understanding of applied sciences like, for example, electrical engineering, mathematics and their relations. Here we compile a few basic general theses from a materialistic oriented theory about relations between mathematics and physics, a field which is in particular considered within the philosophy and history of science. In the following sections we will demonstrate how these ideas blend with the other theoretical approaches we refer to.

In Wahsner and Borzeszkowski (1992) it was claimed that in physics (and we believe: similarly in electrical engineering) "objects" appear as "moving" (or "behaving") and that "moving" is considered in certain dualistic forms like geometry and dynamics or particle and field, which express the categorical unit of "object" and "behavior". The dualistic forms root in the necessity that measuring, and physics implicates to measure, do not only require to differentiate moments but to separate them. In contrast to physics, mathematical "objects" appear merely as "position" within functionally structured systems that presuppose their existence. In other words, mathematical "objects" are subsumed under the "as relation thought property" (p. 133). This handling of "objects" is the specific characteristic form, which allows mathematics to be without inherent contradictions, but, at the same time, disable mathematics to make assertions about real "movements". Consequently, mathematics needs physics (or another empirical science) to be applied to "moving". On the other hand, physics needs mathematics for measurements, calculations and formulating dynamic interactions by laws.

In philosophical and concrete historical studies Wahsner and Borzeszkowski (1992, 2012) figure out those conceptual and experimental-objective preparations within physics that facilitate to use mathematics as mean for recognizing real "moving" and to link mathematics with "measuring". The following both aspects are in particular important:

1. Since only "finite distances" are measurable, (conceptual) contradictory identifications of infinite or infinitesimal quantities, which arise in modern mathematical structures, with finite quantities are enforced. In context specific derivations of model equations, e.g. the heat equation, or relations like the Gaussian theorem, infinitesimal quantities are typically replaced by or used as sufficiently small but finite quantities, which leads to (in a strong logical sense) inconsistent argumentations. The particular context dependent adequate inconsistent use of mathematical concepts is historically one of the most original achievements of physics.
2. Only effects of properties of objects are measurable and not dynamic interactional relations. This leads to the question, which behavior can be transformed to a property. Related answers could be found studying the complicated historical genesis of physical measured quantity.

Whereas physics expresses basic possibilities of nature, theories in engineering sciences represent concretizations in view of subject related aims and embedded within culture-historical as well as socio-economical processes, which implicates that generally the sketched ideas remain relevant.

Within our signal analysis context, a) and b) might be related to the following observations: The use of the Dirac impulse is not only accompanied by the visual representation as an infinitesimal narrow, in ^{R} the support is in fact$\{{}^{0}\}$, and unbounded and infinite "high" function, but also as a function with sufficiently narrow support and sufficiently high values near^{0}. In principle, the latter functions, resp. "impulses", can technically be realized and for those functions the given integral representation is reasonable. Within the electrotechnical context this argumentation is "sufficient", since the "limit" Dirac impulse does not represent a realizable impulse. Analogous arguments apply to the commutation of function limits and differentiation.

In other words, signal analysis considers "mathematical objects" also as electrotechnical quantities, which allows to operate with those objects in argumentations and calculations in a, as such, mathematically inconsistent but electrotechnical adequate way. Of course, the mentioned inconsistencies do not have to become conscious, although they are inherently enclosed in teaching, learning and problem solving activities.

In contrast to the societal aspects sketched later in ATD and by elements of Bernstein's discourse analysis, the moments highlighted within the epistemological-philosophical point might be understood as reified results of historical specific societal processes and as objective and (in a certain sense) logic prerequisites for those in some sense "actual" societal aspects uncovered by ATD and Bernstein. It should be clear that these differentiations are of analytical nature: Neither the epistemological-philosophical highlighted aspects would "exist" if they were not realized and constituted in actual institutional practices nor could the institutionally established practices be understood without the involved historically reified mathematical and electrotechnical concepts together with their basic inner relations.

From our point of view epistemological-philosophical considerations contribute to a deeper understanding of differences or even contradictions between different mathematical practices. The indicated somehow contradictory (or "dialectic") relations between different realized possibilities are constituted within discourses related to different institutional contexts: The mathematical use is addressed in higher mathematics, and the context specific use is addressed in engineering courses such as signal analysis. Here, ATD comes in, since it allows describing and analyzing certain crucial aspects of practices in different institutional contexts.

ATD Analysis

ATD (Chevallard, 1992, 1999; Winslow, Barquero, Vleeschouwer & Hardy 2014) aims at a precise description of knowledge and its epistemic constitution. Behind this approach is the conviction that a cognitive-oriented approach tends to misinterpret contextual or "institutional" aspects as personal dispositions. This aim fits perfectly to the subject scientific approach.

A basic concept of ATD are praxeologies, which are represented in so called "4T-models (T,τ,θ,Θ)" consisting of a practical and a theoretical block. The practical block (know how, "doing math") includes the type of task (T) and the relevant solving techniques (τ). The theoretical block (knowledge block, discourse necessary for interpreting and justifying the practical block, "spoken surround") covers the technology (θ) explaining and justifying the used technique and the theory justifying the underlying technology (Θ). Finally ATD ends up with local and regional mathematical organizations that allow contrasting and integrating practical and "epistemological" aspects of mathematical objects in view of different "institutional" contexts. ATD is in particular helpful in analyzing mathematical knowledge and its different institutional realizations within different learning contexts. This expectation is supported, e.g., by related but differently focusing ATD analyses in (Castela and Romo Vázquez, 2011) considering teaching signal analysis topics in mathematics and two control theory courses.

In terms of ATD the related task T in our example under consideration is to introduce some kind of derivative to a non-differentiable function, which is exemplarily considered by means of the Heaviside function. Applying the sequence of differentiable functions etc. could be seen as τ (technique), "justifying" the limit of the infinitely narrow and infinitely high functions or impulses resp. via aspects of the concept image of the "limit"- and the "integration"-discourses represent θ (technology, discourse). This kind of plausibility arguments is rather typical and important in the context of mathematical argumentations within the electrotechnical discourse. As theory Θ there is a mostly self-contained electrotechnical distribution theory that differs from the mathematical one, for example in its discursive structure. Facets of this electrotechnical theory are related to graphical visualizations that are connected with the effect idea of integration and in particular symbolically based "analogies" to former learned mathematics.

The crucial point is that the signal analysis technique τ does not fit with higher mathematical discourses (technologies). With the epistemological-philosophical considerations as background the electrotechnical technique and discourse has to be qualified as inconsistent but fits in its plausibility and efficiency perfect with the electrotechnical phenomena under consideration.

Applying Elements of Bernstein's Theory

According to our observations about the epistemological-philosophical background and the ATD analysis of an introduction of the delta-distribution, the students have somehow to accept a certain type of inconsistencies and to "learn" that they should neglect specific aspects from those discourses, for example they have to ignore concept definitions (The Riemannian integration of the Dirac impulse is not possible etc.) and at the same time, they have to realize aspects from them, for example specific parts of the "concept image" of integration and limits. In other, (Bernstein's, 1996) words: The students have to understand the context specific principles of knowledge classification (recognition rules) and to apply correctly related "realization rules" in view of solving tasks. For arguments for the relevance of Bernstein's theory in the context of mathematics in engineering courses we refer to Jablonka, Ashjari and Bergsten (2012) and for its general relevance for describing and analyzing teaching we refer to Gellert and Sertl (2012).

Our further empirical investigations show that sometimes tasks send "wrong" signals, that is, they look like mathematical tasks from higher mathematical courses for engineers. Novices in the field then try to apply arguments and techniques from the mathematical discourse but often fail in solving the tasks because of the arising complexity. It requires time and experiences until the students recognize that the electrotechnical discourse establishes some different but subject dependent more efficient techniques (realization rules) that leads to more satisfactorily results. Recognition and realization rules are obviously in strong relation to the selection of premises as well as to contents and structure of reasoning processes. With this observation a link back to our subject scientific approach is established.

Summary

We presented a few observations obtained by considering a specific mathematical topic in signal analysis by three different lenses. We explained their principal usefulness and in particular their compatibility with a subject scientific approach allowing the description and analyzing of learning experiences in view of historical specific societal mediations. Our main observation is that the three applied lenses figure out substantial aspects of the mathematical-electrotechnical subject matter in such a way, that they can be injected as facets of meanings within the subject scientific approach. In particular, they can be seen as generalized societal action possibilities, which, among others, were potentially reflected in subject related reasoning schemes as premises. This kind of meaning exploration has to been seen as a non-trivial and essential first step within subject scientific research. In terms applied in Roth and Radford (2011) and taken in particular form Leontjev and others, these considerations are part of an analysis of "activity" in the sense of "Tätigkeit". Within the dialectic of "action" and "activity" it concerns the societal side. Within a subject-scientific embedding cognitive and affective-emotional aspects have to be understood in an integrated way, such that their non-separable dialectic interrelation will be respected. In other words, premises or actions could not be understood as the result of a cognitive recognition and realization of meanings alone. This was also strongly emphasized by Roth and Radford and in detail illustrated in their empirical investigations. We expect that this is also true in our context where the mathematical object is more complex and the institutional embedding is rather different. We also expect that all three lenses will contribute to further analyses although in different ways. ATD analyses will inform microanalyses of task solution processes and contribute to answers on questions about concrete institutional techniques and their justification. Elements of Bernstein's theory could inform the analysis of specific selection processes between discourse possibilities and the difficulties students will have in recognizing whether a task has to be understood as a mathematical or an electrotechnical one. The philosophical-epistemological considerations could inform a deeper analysis of the relations between these both discourses. Hence these approaches allow unfolding different aspects of the meaning of mathematics in signal analysis. They are relevant for describing and analyzing related "activities" and appear to some extent as the medium, within subjective action reasoning grounds. They represent action alternatives but do not determine them, since there is an "active" unconscious-conscious step of selecting, neglecting or highlighting meanings in view of the subjectively noticed "life interests".

In a next step one has to reconstruct typical premises-reasons-connections as part of the general causes-reasons-connections, which will enclose concrete empirical research questions regarding tasks and solution processes and where among others video and interview data will be used. It is clear that the presented approaches have also potencies for the reconstruction of the "subjective" side of premises: ATD with respect to previous knowledge, which influence for, e.g., how deep meanings can be recognized; Bernstein's theory with respect to predominant accessible discourses and their recognition and realization rules; epistemological-philosophical analyses with respect to the general "world view", e.g. the mediation by societal work considering the cultural-historical nature of knowledge.

Acknowledgement

This research was supported by BMBF 01PK11021D. The authors are grateful to all the colleagues involved in the project KoM@ING for stimulating discussions.

References

Bernstein, B. (1996). *Pedagogy, symbolic control and identity: Theory, research, critique.* London: Taylor & Francis.

Borzeszkowski von, H.-H., & Wahsner, R. (2012). *Das physikalische Prinzip. Der epistemologische Status physikalischer Weltbetrachtung.* Würzburg: Königshausen & Neumann.

Castela, C., & Vázquez, A. R. (2011). Des mathématiques à l'automatique: étude des effets de transposition sur la transformée de Laplace dans la formation des ingénieurs. *Recherches en Didactique des Mathématiques 31*(1), 79–130.

Chevallard, Y. (1992). Fundamental concepts in didactics: Perspectives provided by an anthropological approach. *Recherches en Didactique des Mathématiques, Selected Papers. La Pensée Sauvage, Grenoble*, 131–167.

Chevallard, Y. (1999). L'analyse des practiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques 19*(2), 221–266.

Frey, T., & Bossert, M. (2008). *Signal- und Systemtheorie* (Vol. 2). Wiesbaden: Vieweg+Teubner.

Gellert, U. & Sertl M. (Eds.) (2012). *Zur Soziologie des Unterrichts – Arbeiten mit Basil Bernsteins Theorie des pädagogischen Diskurses.* Weinheim: Beltz Juventa.

Holzkamp, K. (1985). *Grundlegung der psychologie*. Frankfurt/Main: Campus.

Holzkamp, K. (1993). Lernen : *Subjektwissenschaftliche Grundlegung.* Frank-furt/Main: Campus.

Jablonka, E., Ashjari, H., & Bergsten, C. (2012). *Recognizing texts in undergraduate mathematics education.* Paper presented at the 12th International Congress on Mathematical Education. http://www.icme12.org/upload/UpFile2/TSG/0917.pdf

Leontjev, A. N. (1978). *Activity, consciousness, and personality (M. J. Hall Trans.)*. Englewood Cliffs, NJ: Prentice Hall.

Roth, W.-M., & Radford, L. (2011). *A cultural-historical perspective on mathematics teaching and learning*. Rotterdam, The Netherlands: Sense.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Tolman, C. W. (1991). Critical psychology: An overview. In C. W. Tolman & W. Maiers (Eds.), *Critical psychology: Contributions to an historical science of the subject* (pp. 1–22). Cambridge: Cambridge University Press.

Vygotsky, L.S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wahsner, R., & Borzeszkowski von, H.-H. (1992). *Die Wirklichkeit der Physik. Studien zur Idealität und Realität in einer messenden Wissenschaft.* Frankfurt/Main: Peter Lang.

Winsløw, C., Barquero, B., Vleeschouwer, M. de, & Hardy, N. (2014). An institutional approach to university mathematics education: from dual vector spaces to questioning the world. *Research in Mathematics Education, 16*(2), 95–111.