CABRI'S ROLE IN THE TASK OF PROVING WITHIN THE ACTIVITY OF BUILDING PART OF AN AXIOMATIC SYSTEM

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We want to show how we use the software Cabri, in a Geometry class for preservice mathematics teachers, in the process of building part of an axiomatic system of Euclidean Geometry. We will illustrate the type of tasks that engage students to discover the relationship between the steps of a geometric construction and the steps of a formal justification of the related geometric fact to understand the logical development of a proof; understand dependency relationships between properties; generate ideas that can be useful for a proof; produce conjectures that correspond to theorems of the system; and participate in the deductive organization of a set of statements obtained as solution to open-ended problems.

INTRODUCTION

Our research group $\mathcal{Æ} \cdot \mathcal{G}$, constituted in 2003, has centred its activity on issues related to the learning and teaching of proof and proving in Geometry. One of its goals is to identify conditions and actions that foster learning to prove in a university level Geometry course. Particularly, we are interested in using Cabri as a mediating tool in the learning to prove process. Our framework is based on ideas exposed in several research agendas (Radford, 1994; Jones, 2000; Laborde, 2000; Mariotti, 2000; Marrades and Gutiérrez, 2000; Healy, 2000), which promote that when geometric construction tasks are linked with the practice of justifying and organising axiomatic systems, the possibility of learning to prove is increased.

Most research on teaching mathematical proof with Cabri focuses on secondary or high school students in a Geometry course. The principal studies focus on the analysis of the roles of proof in the Mathematics curricula, students' difficulties in proving, and teaching experiments to encourage learning to prove (Mariotti, 2006). Many studies investigate how to introduce the students to a theoretical perspective of Geometry, linking geometrical constructing tasks with production of statements to justify, for example, why certain geometric properties of a construction remain invariant when we drag their free objects (Jones, 2000). Other studies advance towards the teaching of proof, analyzing the role of the drag function in helping students look for properties, special cases, counterexamples, etc., that could be related to form a proof (Marrades and Gutierrez, 2000). But studies based on these same aspects with university students, corresponding to the rigorous treatment required at such a level, are insufficient (Marrades and Gutiérrez, 2000).

In this paper, we show how the software Cabri is used when the activity in the class, with preservice mathematics teachers, consists in building a part of an axiomatic system of Euclidean Geometry following, at least partially, the development of a specific system proposed by a mathematician. The examples hereby presented have been chosen because they help us to show how the students participate in the construction of the axiomatic system, transferring knowledge and information obtained using Cabri, for the justification of geometric facts, to the usual context of a written proof. We will illustrate the type of tasks that impel students to recognize the relationship between the steps of the construction and the steps of a formal argument and thus help them understand the logical development of a proof; understand dependency relationships between properties; generate ideas that can be useful for a proof; produce conjectures that correspond to theorems of the system; and participate in the deductive organization of a set of statements obtained as solution to open-ended problems. With our proposal, we hope to contribute new elements about the use of Cabri in the learning to prove, when the task is building an axiomatic system.

RESEARCH FRAMEWORK

We adopt the sociocultural perspective that views learning as "*becoming a participant in certain distinct activities* rather than as becoming a possessor of generalized, context – independent conceptual schemes" (Sfard, 2002, p. 23). What is learnt, in this case, is a distinctive task of the mathematics community, proving, which includes not only actions related to the act of justifying but also actions associated with formulating conjectures, all of which must be theoretically warranted by an axiomatic system.

About the teaching of proof, unlike a direct axiomatic presentation of a system, we are proposing what de Villiers (2004) denominates "rebuilding approach". The content isn't displayed to the students as a finished structure; it is constructed by the apprentice, with teacher scaffolding, trying to create a typical organization. This approach is promoted by researchers like Polya and Freudenthal (cited in de Villiers, 2004) when they declare that students must follow a similar way by which the mathematical content was discovered or invented. A "rebuilding approach" allows a meaningful approximation to the content and creates the conditions that enable students to actively participate in the construction and development of the axiomatic system.

With the purpose of using the "rebuilding approach" of a part of an axiomatic Geometry system, Cabri assumes a central role, as environment that offers the mediation for the construction of meaning of statements that could be theorems of the theoretical system. Specifically, the main ideas that underlie the design of the empirical study and the analysis of data are:

- The possibility of establishing a correspondence between the figure construction tools in Cabri and the properties and geometric relations of the figures constructed in a classic Euclidean Geometry. This allows the introduction of a method of validation, derived from the analysis of the construction process, and to link the steps of a construction that illustrates a theorem with the steps of a deductive argumentation to prove it (Mariotti, 1997).

- The formulation of open-ended problems that give rise to the production of conjectures of the form *if... then...* as a way to take advantage of the diverse exploration tools that the software has. Radford (1994) proposes modifying the theorems that are going to be incorporated in the axiomatic system into statements of the type: conditions that a certain figure must fulfil so that it has such property, thus creating open-ended problems.
- The possibilities of exploring figures in Cabri with the intention of finding properties that help the students elaborate proofs. The use of Cabri helps students look for properties, create auxiliary lines, recognise parts of special triangles or quadrilaterals that could be linked to form a proof (Marrades and Gutierrez, 2000; Laborde, 2000).
- The "soft" or "robust" constructions (Healy, 2000) that lead to the production of diverse conjectures associated to a family of figures which can be organised to form part of the axiomatic system constructed.

THE TEACHING EXPERIMENT

The sample

The teaching experiment has been carried out, during successive semesters, with future secondary school Mathematics teachers in the course *Plane Geometry*, which corresponds to a second semester course of the curriculum. During 10 semesters, the curriculum blends the study of Mathematics with courses in Mathematics Pedagogy and Didactics. The Mathematics courses cover topics of the principal branches of this discipline: Algebra, Geometry, Calculus and Statistics (more or less with the same requirements as expected when the degree is in Mathematics). The study of didactics and pedagogy is centred on the process of teaching and learning Mathematics and on analysing the Math studied in schools. *Plane Geometry* is the second Geometry course that the students take at the University. The first course, *Elements of Geometry*, has been designed to introduce students to the field of Geometry, using an intuitive and informal approach, where the main tools used are ruler and compass. The aim of this course is to help students gain a conceptual frame for future courses; students analyse several important, but isolated, geometrical properties and work on developing skills such as visualisation, conjecturing, communicating and arguing. It is in *Plane Geometry* where they first face the task of formal proving, within an axiomatic system.

The experiment

The teaching experiment has taken place during the 16 weeks of the *Plane Geometry* course, for several semesters. The topics officially included in the course are the usual ones: relations between points, straight lines, planes, angles, properties of triangles, quadrilaterals, circles, and congruency and similarity relations. Some pre-established conditions for the course are: the general task for both teacher and students is the construction of a portion of an axiomatic system of Plane Geometry, through their participation in problem solving tasks and their social interaction; opportunities for the students to engage in the activity of proving are given since the beginning of the course; the teacher's role is to introduce the students into the activity of proving; Cabri is used as a mediation tool that contributes to form a suitable environment for proving.

We have the conviction that when students explore problems with Cabri, they feel confident about the truth of their conjecture, and find important ideas to help them construct a proof. Also, we are looking for a meaningful approach to the concepts and relations studied, in an environment in which students have the opportunity to work together: (i) exploring geometrical properties; (ii) finding regularities while they solve problems; (iii) making conjectures; (iv) formulating justifications about geometrical facts and, (v) organising those ideas and justifications in a particular axiomatic system. Instead of having the teacher expose the axiomatic system directly, we want the students to make connections between empirical and theoretical forms of working and, to participate as a community, whose task is learning to prove while building an axiomatic system for Plane Geometry, reason why not always the entire course topics are covered during the semester.

The course has always been taught by one of the authors of this paper. One of the researchers was present in all the class sessions during the first semester of 2004, taking field notes and making an audio register of the general discussions and the group work, which were later transcribed. During the successive semesters, the events in the classroom continued being object of analysis and revaluation. The team kept on meeting periodically to decide what events of those classes should be registered and analysed, and to design the tasks to be used. Designing open-ended problems that are related to the axiomatic system so far constructed and are useful for the activities of conjecturing and proving, has been a complex task, in spite of the many beautiful problems that exist in dynamic geometry but whose proofs require geometric knowledge which is far beyond that which is included in our system. The study of all the fragments gave rise to the identification of the examples that are hereby reported to illustrate how we use the software to help build an axiomatic system.

CABRI'S ROLE AND SOME STUDENT RESULTS

Using Cabri to understand the logical development of a proof

One of the main norms established in the class, with respect to the type of proof accepted, is the logically organized argument using definitions, axioms and theorems previously known and accepted by all. To help the students understand the logical development of a proof we use the idea, raised by Mariotti (1997), concerning the relationship between the theoretical process of a proof and the organization required for the construction of a figure that illustrates the theorem.

Statement	Justification	Construction in Cabri and related steps in the proof
1. *A*, *B*, and *C* are non collinear points.	Given	Draw three non-collinear points *A*, *B*, and *C*. (1)
2. $\overleftrightarrow{AB}$ exists.	Line Postulate	
3. $\overline{AB}$ exists.	Definition of segment	Draw $\overline{AB}$. (3)
4. Let *M* be the midpoint of $\overline{AB}$.	Midpoint Theorem	Find midpoint *M* of $\overline{AB}$. (4)
5. $\overleftrightarrow{CM}$ exists.	Line Postulate	Draw $\overleftrightarrow{CM}$. (5)
6. $CM = r$, $r > 0$.	Distance Postulate	
7. Let 0 and *r* be coordinates of *C* and *M*, respectively.	Ruler Placement Postulate	Using compass, circle or measure transfer (requires finding the length of $\overline{CM}$ directly or indirectly (6)), find *D* on $\overrightarrow{CM}$.(8)
8. Let *D* be point on $\overrightarrow{CM}$ such that coordinate of *D* is 2r.	Ruler Postulate	Verify that *M* is midpoint of $\overline{CD}$. (10, 12)
9. $0 < r < 2r$.	Property of real numbers	
10. *C*-*M*-*D*.	First Betweeness Theorem	
11. $CM = \vert r - 0 \vert = r$, $DM = \vert 2r - r \vert = r$.	Ruler Postulate	
12. $CM = DM$.	Transitive Property	
13. *M* is midpoint of $\overline{CD}$.	Midpoint definition	
14. $\overline{AB}$ and $\overline{CD}$ bisect each other.	Definition of bisector	

Figure 1

For example, after the first postulates, definitions and theorems of the axiomatic system are established, we proposed the following problem: *Given three non collinear points A, B and C, show that there exists a point D such that* $\overline{AB}$ *and* $\overline{CD}$ *bisect each other*. All students did a similar construction in Cabri, as is described by a group of students, which we have transcribed as the third column in the table (Figure 1). The teacher then asked them to compare the steps of the construction with the statements and justifications of a proof, which led them to include the proof step number (given in parenthesis) after each sentence and which helped them understand the connections between the proof and the construction.

Using Cabri to help students develop ideas for a proof

The following example, which was designed following suggestions given by Radford, and Marrades and Gutierrez, illustrates how interaction with Cabri, in the process of studying an open-ended problem, provides information that is useful for a proof. This experience took place when the students had finished studying the topics related to triangles and quadrilaterals. The problem we asked them to solve was: *"In isosceles triangle ABC, determine the position of the point P, on the base of the triangle, so that the sum of the distances from P to the congruent sides of the triangle is a minimum. Justify your answer."*

The students started the exploration after constructing the isosceles triangle, locating a point *P* on the base, constructing the perpendicular segments from *P* to the congruent sides and calculating their lengths. They dragged point *P* and very soon realised that the sum is invariant. They wrote conjectures such as: *"It doesn't matter where the point is; the sum of the distances is constant"* (Figure 2).

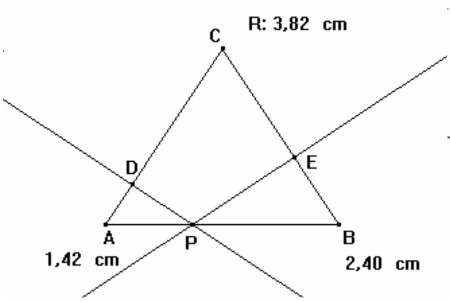


Figure 2

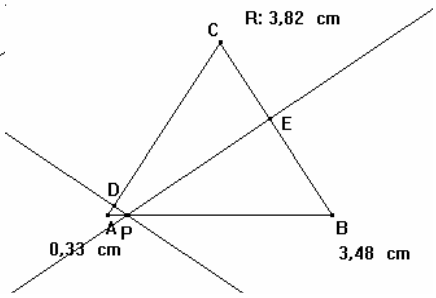


Figure 3

Some students moved *P* until it coincided with point *A* and noticed that $\overline{PE}$ became the altitude of the triangle relative to $\overline{BC}$ (Figure 3). The exploration of a "limit case", locating *P* on one of the endpoints of the segments, shows that they were looking for ideas, based on critical situations, to support their conjecture. The above discovery was very important because it gave them a geometric reference for the sum. It wasn't only a constant value but a very special value: the height relative to the congruent sides. The students then began to draw auxiliary lines, searching for a way to obtain triangles or quadrilaterals, which could be used to prove why the sum is equal to the height, because the elements used in prior deductive proofs had been corresponding parts of congruent triangles or properties of special quadrilaterals.

After some exploration, a group of students discovered how to make good use of the parallelism between the altitude to $\overline{AC}$ and $\overline{PD}$ (Figure 4a). They constructed $\overline{PQ}$

perpendicular to $\overline{BG}$ which led them to the congruency of $\overline{DP}$ and $\overline{GQ}$. This was definitely the key step to be able to prove that $\overline{QB}$ was congruent to $\overline{PF}$, which follows from the congruency of ΔPQB and ΔBFP (Figure 4b).

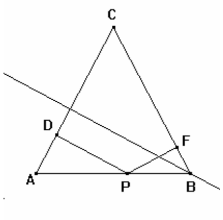


Figure 4a

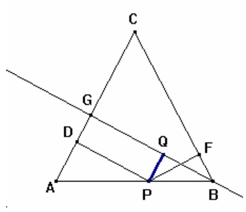
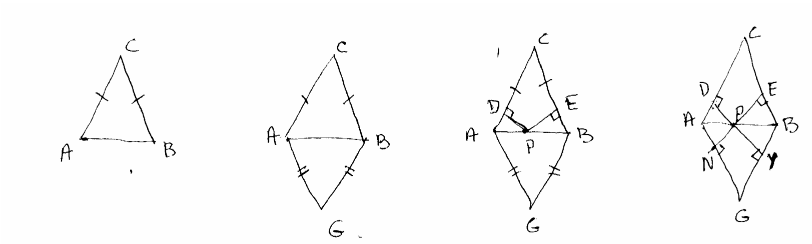


Figure 4b

Another group of students used the symmetry tool of Cabri, which corresponds to a concept not included in the constructed axiomatic system, to reflect the triangle with respect to its base. A careful exploration of the resulting figure, dragging points, taking measurements, helped them realize that the reflected image was congruent to the original triangle. Eventually, this led to the construction of a proof, which one of the students presented to the class, using the following sequence of figures which he drew on the blackboard. The idea underlying their proof is that quadrilateral ACBG is a parallelogram and therefore the distance between opposite sides is always the same, and that ΔDAP ≅ ΔNAP, so DP + PE = NP + PE.



(a) (b) (c) (d)

Figure 5

Using Cabri to create situations where students obtain enough results to collectively organize them as a part of an axiomatic system

For this task, the teacher posed a problem, chosen because of the amount of conjectures students can produce which are related to the situation involved. Using geometric open-ended problems, whose solution permits diverse conjectures about a specific theme, with the support given by Cabri to explore, conjecture and verify results, has shown itself to be a way to involve students in the activity de Villiers (1986) has denominated as descriptive axiomatization.

After formulating a set of geometric properties and relations, as conjectures students feel sure about, with the teacher's guidance, the community organizes the results into a part of the axiomatic system. The teacher's role is essential because she has to design the best path to examine each conjecture, avoid circular reasoning, obtain economical definitions, and establish the correct connections between the results, looking for mathematical coherence. We will report two instances of such situations.

With respect to triangle properties, sometimes students don't understand the need to prove "evident" propositions as, for example, the Isosceles Triangle Theorem: *The base angles of an isosceles triangle are congruent*. We use Cabri to explore interesting properties that require the "evident" theorems, as a means to incorporate them into our axiomatic system. For example, instead of asking the students to prove the above theorem, we ask them to solve the following problems:

> Draw ΔMOP. K is a point on $\overleftrightarrow{MP}$. (a) When is $m\angle OKP > m\angle OMK$? (b) In ΔOKP, when is $OK > OP$?
>
> What is the relationship between the type of triangle and the property: two congruent altitudes?

When students explore these problems with Cabri they find that: (i) the external angle of a triangle is larger than the internal nonadjacent angles; (ii) when two sides of a triangle aren't congruent, then the longest side is opposite to the largest angle; (iii) when two sides of a triangle aren't congruent, then the largest angle is opposite to the longest side; (iv) two of the altitudes of an isosceles triangle are congruent.

Statements (i) and (iv) are properties that students feel they can prove, but when they try to prove the latter, they realise they need the Isosceles Triangle Theorem. Thus we create the necessity of formally including it in the axiomatic system. They also need two new triangle congruency criteria: SAA Congruency (side–angle–angle) and HL (hypotenuse– leg). These criteria can be proved using result (i). The path followed to construct this part of the theory is: Isosceles Triangle Theorem → (i) → SAA congruency criteria → HL congruency criteria → (iv). Another sequence followed is: Isosceles Triangle Theorem → (ii) → (iii). The students' experimental results are organized in a deductive way.

With respect to quadrilateral properties, the problem we use, is the following one:

> What is the relation between the type of quadrilateral and the property: a diagonal bisects the other one?

This is an open-ended problem without a single answer. There are a lot of quadrilaterals which have that property, in a strict sense, and they aren't special quadrilaterals. However, students, unconsciously or deliberately, add other properties to the given one, giving rise to a variety of answers. If students explore the situation using a soft construction (Healy, 2000) and centre their attention on having the diagonal satisfy the given condition and another one, they will formulate conjectures of the form: "if the diagonals of a quadrilateral are… then the quadrilateral is…". They can, for example, imagine that both diagonals bisect each other, and therefore create a parallelogram. Conjectures, like the following ones, are established: "*If a diagonal of a quadrilateral bisects the other diagonal, and both of them are* *perpendicular, then the quadrilateral is a kite*"; "*If both diagonals bisect each other and they form right angles, the quadrilateral is a rhombus*".

But this problem also gives rise to another set of different conjectures. If they begin their exploration, using a robust construction (Healy, 2000) and, following their intuition, construct a special type of quadrilateral to examine the relationship between the diagonals, they will state conjectures of the form: "If a quadrilateral is … then the diagonals…". For example, "*if a quadrilateral is a parallelogram, its diagonals bisect each other*", conjecture that becomes Theorem 1 of the chain that includes all the parallelogram properties which they have discovered or arise in their attempt to prove the conjecture. The axiomatic deductive approach usually employed to introduce the content of the class, changes. The teacher decides which conjecture should be examined first to begin the deductive chain, incorporating other conjectures.

Using Cabri to help students understand dependency relationships between properties

In accordance with Laborde (2000) and Jones (2000), when students explore open-ended problems and write conjectures, they can have difficulty in recognising the properties used in their constructions that conform the "real" hypothesis of their conjecture and therefore guarantee the property discovered. They then write conjectures that don't correspond with the construction that they have made. When students are asked to review the construction process, explain their procedure, we help them grasp all the conditions exposed in the problem, realise whether they have imposed additional or restrictive properties, and understand the dependency relationships involved and, therefore, the logic behind a statement of the form if… then…

For example, when James, a student, was solving the problem related to quadrilateral properties, mentioned above, he wrote the following conjecture: *"In a quadrilateral, if a diagonal bisects the other diagonal, then the quadrilateral is a parallelogram"*. Only when teacher asked him to review the construction did he understand that he was using a more restrictive property, because he included the condition that both diagonals bisect each other.

FINAL REFLECTIONS

Constructing an axiomatic system means not only studying the different elements that conform it: definitions, axioms and theorems, but also understanding how, legally, the latter elements are incorporated into the system, through valid proofs. Being able to construct a proof requires the comprehension of the dependency relations between geometric properties, the ability to visualise auxiliary constructions that permit connections with known facts, the conviction that proving is the only legitimate way to include geometric facts in the system, and the genuine desire to carry on the task.

Students' participation in the proving process increases when they are encouraged to propose new ideas, make conjectures, and listen to and participate in the mathematical arguments of their partners as members of an inquiry community of practice. The teacher has the responsibility to design interesting tasks to promote the mathematical activity of his or her students, establish several opportunities for proving and stimulate a rich interaction so students can move from a peripheral place in the community to the core of it. The use of Cabri to explore open-ended problems allows students to take active part in discovering geometric facts by themselves, and incorporating them, and those discovered in the process of proving the conjectures, into an axiomatic system.

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