INTELLECTUAL NEED AND PROBLEM-FREE ACTIVITY IN THE MATHEMATICS CLASSROOM

Evan Fuller[1]
Montclair State University
Jeffrey M. Rabin
University of California, San Diego
Guershon Harel
University of California, San Diego

ABSTRACT

Intellectual need, a key part of the DNR theoretical framework, is posited to be necessary for significant learning to occur. This paper provides a theoretical examination of intellectual need and its absence in mathematics classrooms. Although this is not an empirical study, we use data from observed high school algebra classrooms to illustrate four categories of activity students engage in while feeling little or no intellectual need. We present multiple examples for each category in order to draw out different nuances of the activity, and we contrast the observed situations with ones that would provide various types of intellectual need. Finally, we offer general suggestions for teaching with intellectual need.

Keywords: Problem solving, DNR-based instruction, High school teaching, Algebra.

[1] fullere@montclair.edu

Years of experience with schools have left us with a strong impression that most students, even those who are eager to succeed in school, feel intellectually aimless in mathematics classes because we—teachers—fail to help them realize an *intellectual need* for what we intend to teach them. The goal of this paper is to define *intellectual need* and explore some of the criteria for an absence of intellectual need. These criteria have emerged from reflection on our observations of classrooms. We will present and categorize classroom activities that we observed in which intellectual need was mostly absent.

The notion of *intellectual need* resides in a theoretical framework called *DNR-based instruction in mathematics* (for more detailed discussions of DNR, see Harel, 2007, 2008a, 2008b, 2008c), although similar notions occur in other frameworks. Harel presents DNR as a system consisting of three categories of constructs: *premises:* explicit assumptions underlying the DNR concepts and claims, *concepts:* constructs defined and oriented within these premises*, and instructional principles:* claims about the potential effect of teaching actions on student learning*.* The initials D*,* N*,* and R stand for the three foundational instructional principles of the framework: *Duality, Necessity, and Repeated reasoning.* Relevant to this paper is the Necessity Principle, which states:

> *For students to learn what we intend to teach them, they must have a need for it, where 'need' means intellectual need, not social or economic need (Harel, 2008b).*

Within DNR, learning is driven by exposure to problematic situations that result in a learner experiencing perturbation, or disequilibrium in the Piagetian sense. The drive to resolve these perturbations has both psychological and intellectual components. The psychological components pertain to the learner's *motivation,* whereas the intellectual components pertain to *epistemology*: the structure of the knowledge domain in question, both for the learner as an individual and as the domain developed historically and is viewed by experts today. At present, DNR is primarily concerned with the intellectual components of perturbations, as emphasized in the Necessity Principle. In particular, the Necessity Principle implies that it is useful for individuals to experience intellectual perturbations that are similar to those that resulted in the discovery of new knowledge. At this point, historical and epistemological analyses have identified five categories of intellectual need in mathematics (Harel, 2008b):

- The *need for certainty* is the need to prove, to remove doubts. One's certainty is achieved when one determines—by whatever means he or she deems appropriate—that an assertion is true. Truth alone, however, may not be the only need of an individual, who may also strive to explain *why* the assertion is true.
- The *need for causality* is the need to explain—to determine a cause of a phenomenon, to understand what makes a phenomenon the way it is. For example, it is arguable (and has been argued historically) that proof by contradiction does not explain what makes an assertion true. Thus, one might continue to experience a need for causality regarding some assertion even after seeing an indirect proof that provided certainty.
- The *need for computation* includes the need to quantify and to calculate values of quantities and relations among them. It also includes the need to find more efficient computational methods, such as one might need to extend computations to larger numbers in a reasonable "running time."
- The *need for communication* includes the need to persuade others than an assertion is true. It also includes the need to establish common definitions, notations, and conventions, and to describe mathematical objects unambiguously.
- The *need for connection and structure* includes the need to organize knowledge learned into a structure, to identify similarities and analogies, to extend and generalize, and to determine unifying principles and axiomatic foundations.

The need for causality does not refer to physical causality in some real-world situation being mathematically modeled, but to logical causality (explanation, mechanism) within the mathematics itself. The need for computation is not a student's psychological motivation to solve drill exercises on algorithms, but her intellectual recognition that realistic and compelling problems *require* the development of efficient computational methods for their solution. These five needs have driven the historical development of mathematics and characterize the organization and practice of the subject today (Harel, 2008b). DNR-based instruction is structured so that these same needs drive student learning of specific topics *and* help them construct a global understanding of the epistemology of mathematics as a discipline. Some intellectual needs have been recognized in other theoretical frameworks under other names. For example, Realistic Mathematics Education (Gravemeijer, 1994) recognizes several specific goals for the core activity of *mathematizing* which correlate with DNR's intellectual needs: certainty, generality, exactness, and brevity. DNR's Necessity Principle is an analogue of the RME dictum that students must engage in mathematical activities that are real to them, for which they see a purpose. Initially, this may mean problems arising in the “real” (non-mathematical) world, but as students progress mathematics becomes part of their world and “self-contained” or “abstract” mathematical problems become equally real. Thus, what stimulates intellectual need depends on the learner at any given time.

Our concern in this paper is to give examples of classroom activities in which intellectual need is absent, and to discuss how the structure of the activity eliminates intellectual need. We explore the implications of DNR for classroom practice by suggesting alternative teaching actions for each example based upon appropriate categories of intellectual need.

When students participate in mathematical activities that stimulate intellectual need, we say that they are engaged in *problem-laden activity*. Unfortunately, many students are engaged in *problem-free activity*, in which they are driven by factors other than intellectual need and, as a result, do not have a clear mental image of the problem that is being solved, or indeed an understanding that *any* intellectual problem is being solved. The idea of problem-free activity can be related to Vinner's (1997) concepts of pseudo-conceptual behavior and pseudo-analytical behavior. In contrast to conceptual behavior, which involves thinking about concepts and their relations or logical connections, pseudo-conceptual behavior looks like conceptual behavior but occurs when one applies a surface-level strategy that does not involve control, reflection, or analysis. Similarly, pseudo-analytical behavior can look like analytical behavior, but it involves procedural knowledge and superficial ideas of similarity without analysis or understanding why things work. For instance, a pseudo-analytical solution of the equation $x^2 - 5x + 6 = 2$ is as follows: factor to get $(x - 3) \cdot (x - 2) = 2$, then set $x - 3 = 2$ and $x - 2 = 2$ to obtain the solution set $\{4, 5\}$. A student using such a procedure views the problem as similar to $x^2 - 5x + 6 = 0$ and tries to apply the solution method from this type of equation without considering whether it makes sense in the current problem. Students engaging in pseudo-conceptual and pseudo-analytical behavior need not be aware of it; they are not usually *choosing* to apply a superficial strategy that eliminates the need to intellectually engage with a topic or problem, even though the behavior has this effect.

Our notion of problem-free activity is similar to pseudo-analytical and pseudo-conceptual behavior in that all involve thought processes that may be difficult for an observer to detect, and there is significant overlap between the phenomena. However, problem-free activity focuses on students' understanding of a task and the needs it might stimulate, whereas pseudo-analytical behavior focuses on the thought processes that are taking place during the activity. It is possible for a student to fully understand and appreciate a problem posed but attempt a pseudo-analytical solution that involves only superficial similarity. Conversely, a student might be unclear about what exactly a question is asking, but nevertheless apply a thoughtful strategy that he understands the justification for. Vinner indicates that it is important for teachers to be aware of pseudo-conceptual and pseudo-analytical behavior, but he does not specify the teacher's role in such behavior. In contrast, the teacher's influence is central to our interest in the phenomenon of problem-free activity.

Mathematical activity falls into a spectrum between extremely problem-free and extremely problem-laden, but many examples fall clearly on one side of the spectrum. For instance, students frequently participate in mathematical activities primarily in order to satisfy the teacher or get a good grade. Teachers may explicitly cite this as the reason for such activity, or they may focus on procedures to the extent that problems become simply opportunities to carry out the procedure that the teacher expects. The tasks that teachers present, what issues are discussed, and the way in which student questions or alternative solutions are addressed all have a pronounced effect on where classroom activity falls in the problem-laden versus problem-free spectrum.

We demonstrate the existence of problem-free activity through numerous examples. The examples are taken from our observations of two high school algebra teachers at a high-performing suburban school, several of whose classes were videotaped and transcribed. We present portions of the transcript with numbered lines, where “T” denotes the teacher and “S” denotes a student. In cases involving more than one identifiable student, numerical suffixes will be used (e.g., S1, S2). When statements are made by multiple students or unidentifiable students, they are denoted by “S*”. Our focus in this paper is not on the particular teachers and students, but rather the phenomenon of problem-free activity and how intellectual need might be restored.

Despite the subjectivity inherent in our analysis--the intellectual need stimulated in a student depends on that student’s interpretations of problems, which we cannot observe directly-- we feel that the evidence for problem-free activity in the episodes presented is quite strong. Four categories of problem-free activity emerged from our analysis and reflection:

1. The situation or immediate goal is not understood by students.
2. The goal of the activity as a whole is unclear.
3. There is no intellectual necessity for the method of solution.
4. Students know in advance what to do, so the problem need never be considered carefully.

In the following sections, we will discuss each of these types of problem-free activity, as well as various teaching actions that promote them. Each category can be manifested in multiple ways, so we present several examples to illustrate its different aspects. Some examples could be included in multiple categories of problem-free activity, but we classify them by the primary type observed. For each example, we suggest alternative teaching actions which could promote appropriate types of intellectual need. This both highlights the aspects of activity that inhibit intellectual need and illustrates some implications of DNR for classroom practice. Finally, we will discuss general recommendations for teacher education that encourage problem-laden activity.

CATEGORIES OF PROBLEM-FREE ACTIVITY

1. Situation or immediate goal not understood by students

In this category of problem-free activity, students do not fully understand the "problem" that has been posed. They may have difficulty interpreting the situation involved, or they may be unclear as to what the goal of the problem is—what question they should try to answer and what a particular answer might mean. This may be due to the wording of the problem (e.g. it doesn't contain a clear question to answer), the level of the knowledge of the students, or other factors. The students we observed generally had difficulty recognizing and checking their answers. They only checked their answer when required to do so, and in such cases the check became an extra prescribed step in the problem itself rather than a control mechanism to evaluate the validity of an answer. We found evidence of many cases in which students did not fully understand what the immediate goal of a problem is or what situation it describes. Such problems will rarely be meaningful for those students. To avoid this occurrence, teachers should emphasize the meaning of problems and provide examples where the goal is discussed before a problem is solved. Asking questions of students such as "how many answers might you expect?" or "how can we see whether this answer works?" can help clarify the meaning of a problem.

Episode 1.1

The following problem was given in class:

A student has a snow-shoveling business, and charges $100 per customer for unlimited shoveling. However, he discounts the price by $1 per customer for each customer over 20. What is the largest amount he can earn?

The following dialogue occurred between the teacher and one of the small working groups that, by means not captured on the videotape, obtained $x = 40$, the correct number of extra customers that will yield maximum earnings.

1. T: If *x* is 40, how do I figure out the amount of money that I made?
2. S: I don't know.
3. T: How do I, if I have 20 customers and I have 100 dollars and I charge them 100 dollars.
4. S: Wait, what? 40 is the extra he got.
5. T: 40 is *x*, what does that mean?
6. S: Yeah, *x* is… I thought *x* was like the extra money… Oh, you subtract it from the customers!
7. T: Right. And so if I…
8. S: That means you have 40 customers?
9. T: 40 extra customers.

In this episode, we find evidence that students acted on a problem without a clear intellectual purpose. The purpose appears to be social: the students introduced the variable "*x*" because that is the first step in the problem-solving approach they have learned. However, they do not appear to have a conception of what the problem is asking for, nor are they clear on what "*x*" represents in context. Despite having found the value of *x*, they cannot answer the question posed by the problem without heavy prompting by the teacher. A student says, "I thought *x* was like the extra money," and the teacher explicitly corrects him, clarifying that $x = 40$ means there are "40 extra customers." The students do not appear to have formed a coherent schema for the problem statement; they do not even know the unit that should be associated with *x*.

A slight alteration in the problem statement might have made it more meaningful to students: asking *How many customers would he want to have?* instead of *What is the largest amount he can earn?*

We would expect some students to initially suggest a large number of customers, perhaps 1,000. The teacher could then ask how much money he would earn with this number of customers. Presumably these students would be quite surprised to find that he would lose $880,000 under these conditions. This surprise could lead to a desire to understand the problem more fully and explore whether he can maximize revenue--and then how many customers would be required to do so. Hopefully, the idea that the student earns very little with a few customers, then earns more for a while, then eventually starts losing money would lead to the suggestion that there should be a maximum, and a *need for causality* in understanding why this is so. Note that such reasoning would implicitly use the concepts of derivative and critical point, as well as the intermediate value theorem. In response to student questions about whether this situation makes sense, the teacher could lead a discussion of mathematical models, which may not approximate reality or even make sense for extreme values of their parameters (does the discount continue for x>100?). After the problem is well-understood and the idea of finding a unique maximum emerges, students could be put into groups to actually find the maximum. At this point, they should be acting from a *need for computation*. Students might guess and check or make use of a table, but in this case it would probably be easier to introduce a variable (which should be clearly defined) and express revenue as a function of this variable. The problem would then lead to a need for a method to find the maximum of a quadratic function; this could be achieved graphically using symmetry or inspection, or algebraically by deriving the formula for the vertex of a parabola.

Episode 1.2

The teacher models how to solve simultaneous linear inequalities, but does not provide a clear problem that is being solved.

1. T: Systems of linear inequalities. So just like we had the equations, right, when we graphed lines, we also graphed inequalities where we shaded a region. With the systems we're going to end up shading a couple of regions and see where those regions overlap. OK, so let's take a look at a couple of examples. So I want to graph $x \geq 2$, and I also want to graph $y < -1$. I want to see where those two intersect and that's going to be my solution. So if I look at $x \geq 2$, first I have to graph $x = 2$. Then I need to look at the region where the x values are greater than 2. OK and again I'm going to have a solid line because of the, I need to include the points that are on the line. Next I'll look at $y < -1$, dashed line because the points on the line are not included, and I want to shade the region where my y values are less than negative one. So where do these overlap?

2. S: The green.
3. T: The green, right. The green part. OK, so this is my solution, right… so when we start working these problems I need you to be able to identify this region… I need you to understand that this is the solution, right? Points up here do not satisfy both inequalities. Points here do not satisfy both inequalities. So you'll need to show me this region.

A method is being presented to do something, but it seems unlikely that students perceive a problem or goal for this activity. They may learn the procedure: when you see two linear inequalities, draw a picture of this kind and try to figure out which region to shade. However, the usefulness of that picture and what exactly it represents are not discussed; nor is the picture compared to an algebraic solution. The teacher's statement, "I need you to understand that this is the solution," presents strong social need (this is important to the teacher) but no intellectual need. Since it is not clear what "this" is the solution *to*, students are probably confused. Nowhere in the presentation is there a clear statement of the purpose of this activity, nor is there any emphasis on how the activity corresponds to a problem.

Obviously, it would be useful to present a problem here, which could be stated as: which points in the plane have *x*-coordinate at least 1 and *y*-coordinate at most -1. Students could be asked to give examples of such points, which they should readily come up with. Then the problem of displaying all of them could be presented—a *need for communication*. This simple problem is then situated within an important mathematical context: how to effectively describe an infinite set. The condition of *x*-coordinate being at least 1 could be related to the point lying on or to the right of the line *x*=1. The condition of *y*-coordinate being at most -1 means that the point must lie below the line *y*=-1. Thus, we are looking for points on or to the right of the line *x*=1 AND below the line *y*=-1. The appropriate region could then be shaded on the graph as a way of presenting this infinite set.

Episode 1.3

The teacher presents an application for what the class has been working on: finding the *x*-intercepts of a parabola.

1. T: How are we going to use this in real life? Mr. Jamison owns a manufacturing company that produces key rings. Last year, he collected data about the number of key rings produced per day

and the corresponding profit. He then modeled the data using the function $(k) = 2k^2 + 12k - 10$, where P is the profit in thousands of dollars and k is the number of the key rings in thousands.

2. T: OK, he is going to make a whole bunch of key rings. He is going to make thousands of key rings, and the number of key rings he makes if he makes ten thousand key rings, then k is equal to what? …
3. S1: Key rings.
4. T: *k* is the number of key rings in thousands. So if he made ten thousand key rings, what is *k* equal to?
5. S2: 10.
6. T: *k* is equal to?
7. S1: 10.
8. T: 10, OK, and *P* is the profit, if he made ten thousand key rings and we plug in 10 for *k*, we'll find out how much money he made. And if it turns out that this number over here is 5, how much money did he make that year?
9. S1: $5000.

It sounds like the teacher is helping students explore the problem in this episode, but in fact no question has been posed—a potential indication of problem-free activity. Moreover, the students are acting on an object--the string of symbols $P(k) = 2k^2 + 12k - 10$—whose purpose is not clear. The teacher later examines how to find the maximum profit using a graph but, in order to necessitate the profit function, this question should be part of the original statement.

This episode continues after the teacher has found that the maximum profit occurs when 3000 "keys" are made:

1. T: If we own a company that is producing keys, for some reason he is going to make more and more profit until he makes 3000 keys, after he makes 3000 keys he is losing money. Now why is that? That could be because his factory is idealized at this and if you try and work people too hard then they don't produce as well. Maybe their quality goes down. Maybe the machinery starts breaking down as they produce more keys.

(….)

2. T: Is he making a profit down here?
3. S*: No.
4. T: No, what happens when it's down here?
5. S*: [inaudible]
6. T: Losing money, and the union went on strike or something so that now he is losing money instead of making

money. OK? So this is his goal to try to get here, but he wants to make sure that he makes a profit. Where is that point on the graph where he is not making money, he is not losing money? It's called a break-even point.

(….)

7. T: Tell me how many keys he has got to make to break even.
8. S: How many keys?
9. T: How many keys, key rings, must he produce?
10. S: 6000.
11. T: 6000, why? I want you to find that mathematically. How many keys, key rings has he got to produce to break even? He just discovered that he has to pay taxes on all his profits, so he doesn't want to make money, he doesn't want to lose money, he wants to be right there where there is no taxes. How do we find that point?

This teacher does help students explore the problem and understand the meaning of the function *P(k)* in context as well as its graph, making connections between the geometric and algebraic representations. However, the problem statement is still not made fully clear. The term "modeled" is not mentioned in the discussion—they take *P(k)* to be the exact profit. The meaning of "corresponding" profit is also not made clear; it is apparently supposed to be daily profit (rather than annual profit as the teacher implies in line 8), but the causal chain from production to sales must be too long for each day's production to determine that day's profit.

The teacher begins to refer to "keys" rather than "key rings," which seems to confuse a student (line 17). After it is found that the maximum profit occurs when three thousand "keys" are produced, the teacher tries to make a connection to the physical reality of the situation by suggesting reasons why the profit decreases beyond that point. Unfortunately, the reasons he gives (people or machinery start breaking down, line 10) are unrealistic; it seems clear that the intended reason for decrease in profits as production goes up is that there will not be enough demand for that many key rings unless the price is lowered. The lack of realistic connections in this problem may have caused it to be viewed as artificial: a way to test what students know rather than a meaningful application of their knowledge. Recall that the teacher brought up the problem by saying "How are we going to use this in real life?"

When a student suggests that 6000 “keys” must be produced to break even, which is an incorrect answer, the teacher asks why. However, the teacher does not provide an opportunity for this student to check or justify his answer. After saying “I want you to find that mathematically,” the teacher continues his unrealistic interpretation of the problem: “He just discovered that he has to pay taxes on all his profits, so he doesn’t want to make money, he doesn’t want to lose money, he wants to be right there where there is no taxes” (line 20). Following the episode quoted, the teacher launches into a discussion of the quadratic formula, which could further support the conclusion that the “problem” was intended to be a chance to practice (among other things) the quadratic formula. Note also that the teacher’s questions seem to call for a single “break-even point” even though the quadratic formula yields two answers (5 and 1), both of which are physically realizable. This conflict is not mentioned in the class.

For this problem, it would be useful to ask from the beginning: how many key rings will Mr. Jamison choose to produce? The discussion of the profit function would then be quite reasonable: we have some data on which to base this decision, so we should make sure we understand it. Finding the profit when 10,000 key rings are produced daily would be a useful exercise. However, the teacher could also ask exactly what “profit” means, to make sure that students understand the idea. He could discuss the act of modeling the data to obtain an equation for profit in terms of daily production and the fact that production must remain constant for a long period in order for daily profit to match up with daily production. The teacher could ask whether or why students expect there to be a unique production level that maximizes profit. Graphing the solution helps to demonstrate what is happening but not why, leading to a *need for causality*. If students thought about the situation for several minutes, some might suggest the issue of supply and demand: that larger numbers of key rings cannot be sold unless the price is lowered. The relevant data for calculating profit would be key rings sold, not key rings produced. It would be reasonable at this level for students to consider how the equation $P(k) = -2k^2 + 12k - 10$ might have been obtained, and why units of 1000 are convenient. That is, in order to sell *k* (thousand) key rings, the price must be set at $-2k + 12$ dollars each, and the fixed cost for running the factory is \$10,000 per day. Once these issues are understood, the teacher could ask what range of production levels of key rings will allow Mr. Jamison to make a profit. This would lead to the issue of finding the boundary points of this range where he breaks even—the *k*-intercepts of the function. Through this *need for computation*, students could realize that *k* in this context plays the role of *x* in the standard quadratic function, and that the intercepts can thus be found by factoring or applying the quadratic formula.

Episode 1.4

Imprecise definitions of variables such as "*b* = base, *h* = height" are quite common in many classrooms. Such imprecision can lead to a failure to accurately represent the problem situation. We see an almost complete lack of referential definitions during this episode, in which one of the researchers (denoted by H) interacts with several students in the class.

Problem: *if Kim can paint a house in 8 hours and Ron can paint it in 4, how long will it take them together?*

1. S1: *x* is Ron and 2*x* is Kim.
2. S2: Yeah, Kim goes twice as slow.
3. S1: Yeah, Kim goes twice as slow as Ron so we did 2x and then…
4. S2: Plus *x*.
5. H: What is *x*?
6. S1: *x* is…
7. S3: *x* is like the, what do you call it?
8. S1: The base.
9. S3: The amount, the amount of time. Like the amount of time it's going to take to, to take to paint the house.

(…)

10. H: So *x* is how long they work together.
11. S1: Yeah.
12. H: OK. So all right.
13. S1: Well *x*, *x* is just how one of them, Kim does it, cause she does it twice as slow as Ron and so 2*x*…

These students associate expressions like *x* and *2x* with the entire problem situation, or with particular characters in it, and cannot say what quantities are being represented. Perhaps because they recognize that their definitions are inadequate, the students change the meaning of "*x*" and "2*x*" several times. From "*x* is Ron," they proceed to "*x* is the base," perhaps a carryover from word problems involving the base of a triangle or rectangle. After some probing, they seem to agree that *x* is how long they work together to build the house. However, they still want to somehow add *x* and *2x*, reverting to the idea that *x* is just how long it would take Ron or Kim alone (which is already known). Throughout the process, these students do not appear to have a goal in mind besides the generic 'write an equation and solve for *x*.' In particular, they do not demonstrate a coherent representation of the problem situation even after exploring what quantities *x* might represent.

Here it would be useful to pause to carefully define any variables that are used to solve a problem so that everyone understands their meaning—satisfying a *need for communication*. Often, as is possible in this case, we define a variable (unknown) for the quantity we are seeking. Thus, it is reasonable to let *x* be the time (in hours) needed for Ron and Kim together to paint the house. However, it is also possible to solve this problem without using variables: by enacting the situation with a diagram or adding the individual rates of painting to find the joint rate, from which the time to paint can be found. Students should be shown that it is useful to understand the problem situation and think about how the question might be answered before introducing one or more variables.

2. Goal of the activity as a whole is unclear

Students may understand what a particular problem is asking, but they often do not perceive an overall goal behind working such problems. When they do not see what they are really learning or what the purpose of such ideas is, students are unlikely to find problems meaningful. Students will often continue working in order to satisfy the teacher, but they occasionally ask about the purpose of activities. In some cases, the teacher may not have an overarching goal for a set of problems beyond practicing for the test, and thus it is impossible for him to satisfactorily answer such questions. Alternatively, a teacher may have a clear overall goal but feel that students need not be aware of it or that asking about the purpose of tasks is a challenge to her authority.

Episode 2.1

One teacher presents tasks that require students to practice skills they should know. Unfortunately, the tasks are pure symbol manipulations that may not represent meaningful problems for the students. Some examples are:

Factor $3x^3 - 9x^2 + 9x$.

$3(x^3 - 3x^2 + 3x)$.

$3x(x^2 - 3x + 3)$.

$3x(x - 3) \cdot (x + 3)$.

$x(3x^2 - 9x + 9)$.

Multiply $(x + 3) \cdot (3x - 9)$.

What is the degree of the polynomial $2x^2 + x^4 + x^3 + x + 5$?

These examples are unlikely to stimulate intellectual necessity: why does one care about being able to factor or multiply isolated polynomials out of context? The directions in the problem may be clear, but the purpose of solving such problems is not. In fact, we never observed this teacher explain the purpose of factoring, which is thus a prime example of an action that students carry out without a clear intellectual purpose.

Spending more time developing the need for multiplying and factoring polynomials before launching into lots of practice problems could be helpful in creating necessity—generally, a *need for computation*. Polynomials come about in many contexts and can be used to approximate essentially any function (though students at this level will not know about Taylor expansions). Thus, it useful to characterize their behavior in several different ways. The long-run behavior depends on the term with the highest power of the variable, thus necessitating the idea of degree. Finding zeros is generally important, and this necessitates both factoring (through the zero-product property) and the quadratic formula.

Episode 2.2

Students are assigned problems to simplify radical expressions, which are in simplest form 'by definition' if they comply with the three laws projected on the screen.

> On Screen: A radical expression is in simplest form when all three statements are true.
>
> (1) The expression under the radical sign has no perfect square factors other than 1.
>
> (2) The denominator does not contain a radical expression.
>
> (3) The expression under the radical sign does not contain a fraction.

The teacher presents very explicit requirements for the form of solutions. However, the task of simplification is not necessitated; there is no discussion of why such a form would be desirable. Only social necessity is provided: the implicit idea that credit will only be given to answers in this simplest form. Once again, the objective of particular problems may be clear, but the purpose of working such problems is not.

The lesson could be presented very differently. Rather than beginning with fixed "laws," the teacher could have assigned several problems whose answers were radical expressions that could be derived in several ways—so that different-looking answers would be likely to arise. Comparing student answers to each other or to an answer in the textbook could lead to a *need for certainty*: the students have to figure out which answers are correct. This need might lead the students to find a method for comparing answers that look different to see whether they have the same value. In this context, the easiest way to compare answers would be to square them and check whether the resulting integers, or fractions put into lowest terms, match. However, students might wish to avoid having to perform this computation to check their answers. The desire to quickly check answers might lead to a *need for communication* whereby a standard form for reporting answers would be developed. With some prompting, students would probably choose a standard form that would obey the teacher's laws: taking out all perfect squares and fractions, and agreeing to put the radical only in the numerator of a fraction. In this way, students would still be able to simplify radical expressions, but they would also understand the purpose of such simplification. Moreover, the process of thinking about such issues should reinforce the idea that radical expressions have values that remain the same even when the form is changed, such as when simplifying. It could also raise deeper questions such as: is the standard form unique?

Special case: students ask for intellectual need, but receive only social need

Episode 2.3

After students work on the snow-shoveling problem (from episode 1.1), the following exchange occurs:

1. S1: Mr. [Teacher], why does this matter?
2. T: What do you mean?
3. S1: Like who cares?
4. T: Well, you care if you want to get a good grade on your test.
5. S1: OK, no, but like in real life?
6. T: Well, we do it because we love math, right? We enjoy a challenge.
7. S2: We love math.

This episode highlights two ways teachers can explain the goals for the material they teach: because it will be on a test, or because it provides a nice challenge. Note that the student explicitly rejects the social necessity to do well on the test as justification for the usefulness of what is being taught, but the teacher does not provide any intellectual necessity for it. In addition to the frustration this answer might cause for students who do not already "love math," the exchange sets up an environment in which students do problems only to get good grades or experience a fun challenge. While these can be motivations for working on problems, they are not the purpose of such problems.

Teachers should make sure that activities they assign have specific goals and that those goals are communicated to the students. Teachers should also be prepared to provide an intellectual answer to the question of why one would care about the particular lesson they are teaching. Here, the teacher could have mentioned that this is a real (modeled) situation raising a practical mathematical question. He could also describe how these problems teach general lessons such as optimization under a condition involving diminishing returns.

3. No intellectual necessity for the method of solution

We expect teachers to help students develop tools that can be used to solve problems. However, the type of tools and ways they can be acquired vary greatly. Students are sometimes given a method for solving a problem along with the problem itself. They may be directly shown how to do all problems of a given type, or they may be told that a problem will use the procedure they just learned. The format for answers may also be strictly prescribed (sometimes arbitrarily), as occurs in episode 3.1.1 below. When the method for solving a problem is not necessitated, the students' task can become applying a method rather than answering a question. Such activity is likely to be problem-free even when the original problem could have been meaningful: students want to resolve the situation, but they have to use the solution method that has recently been taught even though the situation is easily resolvable through other means. For instance, a problem intended to require algebraic manipulation may be easier to solve through guess and check or arithmetic calculation. This category of problem-free activity is important and pervasive. Teachers can and should require that students be able to explain and justify their solutions to the class. However, when students are required (even implicitly) to use only the most recent procedures in solving a problem, that problem loses intellectual legitimacy. Students who were initially interested in the problem may even come to reject it, viewing the task as simply an excuse to practice a method that the teacher says is important. On the other hand, if students are given free reign in solving a problem and then realize through subsequent discussion that the standard approach is preferable to other solution methods, we believe they will be much more likely to appreciate and internalize the standard method.

Episode 3.1.1

The class is working through the following problem:

> When a popular brand of CD player is priced at $300 per unit, a store sells 15 units per week. Each time the price is reduced by $10, the sales

increase by 2 per week. Help the store manager to build a function that gives the total sales revenue for each $10 reduction in the CD player [sic]. What selling price will result in weekly revenues of $7000?

1. T: Right here it says help the store manager build a function that gives the total sales revenue.
2. S: But why does the store manager need a function if he knows the answer?
3. T: Because he wants to be able to substitute any number in for, depending on the week, what he's going to sell.
4. S: He has a table. He can just look at the table. It's much easier than a function.
5. T: What about – No it can't. What if he decides he wants to start at $350?
6. S: Then he can make the table up by $50.
7. T: A function, please. Thank you.
8. S: Well, will you show me how to do the function?

This student does not perceive a purpose behind the teacher's interpretation of the problem: "why does the store manager need a function if he knows the answer"? If an answer is important and it can be found without a function, there is little intellectual need to find a function. The student argues that, for this particular situation, a table would be "much easier" while still providing the needed information. The student is correct that, since a store manager in this situation would probably be interested in a small range of discrete values, a table could provide an efficient representation of the data. Representing data efficiently in a realistic situation is likely to constitute a meaningful problem for students, but the teacher's insistence on finding a function reveals this task to be mostly a front for getting students to use the most recent technique they have learned. The teacher begins by trying to give intellectual reasons for a function: "to substitute any number" (Line 3) and "what if he decides he wants to start at $350? (Line 5). However, when the student argues persuasively that these reasons are inadequate, the teacher turns to purely social need: "A function please. Thank you." In the end, he requires a prescribed form for the answer to a seemingly arbitrary question.

The CD player problem is isomorphic to the snow-shoveling problem (episode 1.1). However, the teacher expects students to use arithmetic sequences and their properties in the solution method for the CD player problem. Sequences had not yet been covered when the snow-shoveling problem was assigned, but they were apparently discussed in the days before this episode, which continues the discussion of the CD player problem.

Episode 3.1.2

1. T: How do you get [pause] how did you get this dollar value [7000]?
2. S1: 300*15, 290*17, [pause]
3. T: Right. You multiplied those two things together, right?
4. S1: Oh.
5. S2: So that's the formula? These are the formulas?
6. T: Well, these are a start for what they look like.
7. S2: But wouldn't this one be…this one…what do you mean?
8. S1: I got it.
9. T: This one's decreasing by $10, right?
10. S2: Mmhmm.
11. T: And then you can simplify these. Can you?
12. S2: Yeah. By like just making it simpler but then [pause] so do we leave *n* in the formula?

In this exchange, the students have already found the answer (a selling price of $200) and explained their reasoning, but the teacher wants them to justify their result using what they know about sequences rather than by working through possibilities. This is a case where the 'problem' has already been solved but students continue working to satisfy the teacher. We see that S2 picks up on the expectation that there will be two formulas—corresponding to the distinct arithmetic sequences that should be used in solving this problem. The students do not appear to perceive an intellectual reason for the teacher's method: they are confused by the instruction to simplify and appear to want to revert to considering specific values of *n* rather than a general formula.

Episode 3.1.3

Later in the same class, the teacher circulates among students that are still working on the CD player problem:

1. S3: What would you use for the common difference? –10 or 2?
2. T: Well, I'm looking at two different sequences here, right? Here's one sequence, here's another sequence.
3. S3: So you're actually writing two equations?
4. T: Mmhmm.

And later:

5. T: No, look at this. Look at this right here. 300, 290, 280, 270. What I want you to do is I want you to generate a formula for this sequence.
6. S4: x-3?
7. T: x-3?
8. S4: Oh, wait, no. For the second one [inaudible] x+2.
9. T: x+2?
10. S4: See, 15+2 is 17, 17+2 is [pause]
11. T: So that's a recursive formula, right? I want the explicit formula. And then do the same thing here. 15, 17, 19 [pause] get me a formula that's going to generate this sequence. OK? So I want to see two formulas. One for this sequence and one for this sequence.

The teacher continues to advocate a single method of solution that has not been necessitated. It appears that these students approached the problem in their own ways, although we do not observe how they attempt it and the teacher never asks for an overall account of student reasoning. Instead, he leads students to follow the method he wishes them to use by prescribing actions they should perform: "300, 290, 280, 270. What I want you to do is I want you to generate a formula for this sequence" (line 5) and "… get me a formula that's going to generate this sequence. OK? So I want to see two formulas" (line 11). The teacher does not say why these actions will lead to the desired solution, nor does he explain clearly to students why their methods will not work or are not appropriate. Overall, it appeared that the task of using the teacher's method correctly was more difficult for students than the task of ascertaining by their own means the selling price that will result in revenues of $7000.

The teacher could have taken the first student's arguments about using a table as an opportunity to compare a formula to a table of values as different ways to give information about a function. Again, it would have been useful to ask where the store manager should set the price, which could create intellectual necessity for a function giving sales revenue in terms of $10 reductions in price. The table might suggest a maximum, but an explicit formula for the function would provide more confidence that the maximum is absolute, (given that parabolas have one vertex) satisfying a *need for certainty*. There will also be a *need for computation* if the optimal price is not a multiple of $10. Once there is a reason to seek a formula, the teacher could aid students who are having trouble by suggesting a connection to finding formulas for arithmetic sequences. However, students should be free to generate the formula for revenue in any way they can.

Episode 3.2

Students are having trouble with the problem, *If it takes a printer 20 minutes to print 400 stickers, how long will it take to print 1100 stickers?,* and the teacher suggests a simpler version. He is at the front of the room addressing the entire class while various students respond.

1. T: If it takes a printer 20 minutes to print 400 stickers, can you tell me how many stickers it could print in one hour? 1200, why?
2. S*: [inaudible]
3. T: You multiply 20 times 3. You recognize that there's 60 minutes in a hour, so 20 minutes is 1/3 of that. You multiply 20 minutes times 3. So what you're talking about, we got a name for that in Math. What do we call that? When two numbers are compared to each other. Three is to four? I got [pause] I've got a portion of a dollar here, right. I've got 1/4 of a dollar. What do we call that? What do we call that 1/4?
4. S*: A quarter.
5. T: You're right. What do we call that comparison of numbers. One to four.
6. S*: Ratio.
7. T: A ratio… Can we make, and by the way we combine two ratios together, that's called a ratio is equaled to something else. What do we call that?
8. S*: Cross-multiply?
9. S*: Fractions?

10. T: Well, they are fractions. We cross-multiply to solve it, but what do we call this, this is called a [pause] It starts with a p? Starts with a pr? Pro-? You're a tough crowd folks, prop, proportion.

(…)

11. T: Now, what did you tell me before when I asked you to figure out how many, how many stickers could be printed in an hour? Why? What was the ratio?
12. S*: [inaudible]
13. T: Yeah, but how did you get that? How did you figure it out? What was the ratio you used?

A student came up with the answer (1200) to the simplified question without using an explicit ratio. The purpose of the discussion seems to be for the students to understand the teacher's reasoning and learn to construct explicit proportions. While the teacher explains the steps he used to find the solution in line 3, he does not construct an explicit proportion. The focus instead seems to be on the conversion factor "60 minutes in an hour," by which 20 minutes becomes 1/3 of an hour. The implicit idea that 3 times as many stickers will be printed in 3 times as long a time interval involves proportional reasoning, but this fact may not be clear to the students. Most of this episode is devoted to naming things and identifying how they appear rather than how proportional reasoning is used. The teacher asks for "the ratio" from the problem twice (lines 11 and 13) without specifying which ratio. Presumably, he has in mind some form of the proportion:

$$\frac{\#\ of\ stickers\ printed\ in\ 20\ min}{\frac{1}{3}hour} = \frac{\#of\ stickers\ printed\ in\ 1\ hour}{1\ hour}.$$

Based on their lack of clear responses, this interpretation does not seem clear to students. Moreover, because the simple problem given is a case for which the teacher's implicit reasoning is easier than constructing a proportion, there is no necessity for using a proportion. After returning to the original problem, the teacher presents the correct proportion: $\frac{20\ min}{x\ min} = \frac{400\ stickers}{1100\ stickers}$. However, he does not devote much attention to the link between informal proportional reasoning and this explicit proportion. His reasoning on the simpler version of the problem would suggest a different way of thinking about the situation: that it will take $1100/400$ times as long to produce $1100/400$ times as many stickers, so the answer is given by

$(1100/400)20\,min = 55\,min$. The problem may have initially stimulated intellectual need, but there appears to have been only social need for using an explicit proportion.

After offering the simpler version of the sticker problem, the teacher could have asked how long it would take to print 800 stickers and gotten the answer from students, then he could have asked students if anything is remaining constant in these situations. This could provide a *need for connection and structure* in that students want to figure out how the simpler problem formulations are related to the original one. Students would probably see that something is the same in these situations, but they would have trouble articulating it. They might at least recognize that some of the same numbers, 20 and 400, are being used. The teacher could point out that such a problem cannot be solved without assuming that the machine is "running at a constant rate" and ask the class for suggestions as to exactly what this should mean. Through discussion, they could realize that the rate of stickers produced should be treated as constant in these situations. This could be formalized into an equality of ratios that holds regardless of what time interval one examines: $\frac{\#\,of\,stickers\,produced}{time} = \frac{\#\,of\,stickers\,produced}{time}$.

At this point, students could be led to see that the left side (for instance) will always be $\frac{400\,stickers}{20\,min}$ and that the right hand side will contain an unknown quantity in either the numerator or denominator. If this equation can be solved for the unknown, the problem will be finished. The teacher could then ask for comparisons between this explicit use of a proportional equation and the informal reasoning applied by both teacher and students. He could also assign further problems involving proportions and suggest that students try them using informal reasoning and explicit use of a proportion to see which they prefer. The informal reasoning may be deemed simpler, but the explicit proportion might be considered more general and (after practice) more clear.

Episode 3.3

> The teacher is leading the class through a solution to a worksheet problem, allowing various students to respond. The class used a mnemonic called LESA [Let the variable stand for something, write an

Equation, Solve the equation, Answer the question] to solve word problems.

1. T: Number 5 says, find two consecutive integers such that the sum of their squares is 61, so “L”, what are our variables? C’mon, folks.
2. S*: 2 and 61. [hard to hear]
3. T: What is it?
4. S*: 2 and 61.
5. T: Are those variables?
6. S*: *A* and *B*.
7. T: What are we looking for?
8. S*: Two integers that go in 61.
9. T: OK, we’re looking for two integers, so I’ll call integer *A* and one integer *B*.

(…)

10. T: OK, so can we come up with two equations for our unknowns?
11. S*: $A^2 + B^2 = 61$
12. T: Everybody agree with that one?
13. S*: I guess.
14. T: OK, what’s the other equation? Do we have another equation here?
15. S*: You get the 61 and bring it to the other side.
16. T: Hold on, I don’t want to solve them. I am looking for a second equation here. I got two unknowns, I need a second equation here.
17. S*: Are you supposed to do like $B + 1$ because it’s like a consecutive number?

In this episode, no intellectual necessity is presented for either introducing variables or finding a second equation. Rather than asking the students for suggestions, the teacher leads the class through a prescribed method without exploring other ways to think about the problem. Students do not object to this; they seem used to following a method without clear goals. For instance, one student suggests “2 and 61” as the variables. We surmise that he is acting based on the social need to find something that can act as a variable in the problem; he chooses the only two numbers present in the problem statement. In this context, the teacher’s question “Are those variables?” is meant to suggest that this response is incorrect, rather than to elicit the student’s understanding of the term “variable.” Other students pick up on this hint and suggest “*A* and *B*” before choosing what the variables are to represent, suggesting that the primary necessity for variables is social necessity. After eliciting the equation $A^2 + B^2 = 61$, the teacher does not discuss why a single equation with two variables cannot be solved. He simply says “I don’t *want* to solve them” (our italics) and “I need a second equation here” (line 16). The teacher could have used this opportunity to necessitate a second equation by helping students realize that they will not get a unique answer without additional constraints. Instead, he appears to follow a fixed rule that a single equation in two variables cannot be solved. The focus in this episode is on correctly applying the LESA approach rather than on necessitating the particular methods that are used to solve this problem.

In order to implement the necessity principle, problems should be difficult enough to warrant the desired solution method. In this case, the problem is simple enough that it could be most easily solved by guess and check or systematically trying integers. A more sophisticated type of reasoning, which the teacher could model, would be to notice that the average of the squares is 30.5, and one must be below the average (25), the other above (36). If the goal is to develop students’ ability to solve systems of equations, then more difficult problems should be given for which students are given the opportunity to guess but are unlikely to do so successfully. Once there is a need to write down equations, then questions concerning how the number of unknowns and number of equations relate to the solution set can be posed.

Generally, the use of variables can be necessitated by a *need for computation* (if the unknown numbers are very large, or not integers), for *communication* (to explain the reasoning used to solve the problem), or *structure* (when systematizing the approach to apply to a class of similar problems).

4. Students know what to do in advance, so the problem need never be considered carefully

Teachers rarely want their students to encounter the full difficulty of a significant real-world problem. Some difficulties must be avoided or minimized, requiring teachers to choose which difficulties to avoid and which to exploit for creating useful disequilibrium. In an effort to make things easier for students, teachers will sometimes give problems in standard forms that invite students to use a known method rather than exploring the meaning of a problem and different ways to solve it. Even when a particular method is not imposed on students, such actions can deprive students of learning opportunities, because it is precisely the difficulties and confusion in solving problems that destabilize students' current knowledge and require them to extend their thinking. When students know what they will do to complete a task before engaging with a problem, they are involved in problem-free activity.

Episode 4.1

The teacher is showing how to find the intercepts of a cubic polynomial in order to graph it. The cubic has an explicit factor of *x*, but he also promises that the remaining quadratic is factorable:

> T: …And I end up with a quadratic inside the parentheses. And in this case I can factor what's inside the parentheses. If I'm not able to factor what's inside the parentheses, we can still find the *x*-intercepts by using the quadratic formula or completing the square, but for right now, for what we're going to start off with, I'm gonna give you equations that are – that we're able to factor in here.
>
> Later in the same discussion:
>
> T: And in fact, on the homework that I give you tonight, I'm not gonna ask you to sketch any graphs, I'm just gonna ask you to solve the equations…And for right now, I'll worry about you guys finding the linear factors for yourself later – for right now, if I give you one that's gonna need a linear factor, I'll just give you a linear factor, and then you can factor the quadratic after you use the factor that I give to you.

Intellectual need is stimulated when a student is led to desire a resolution of a situation—which is unlikely to happen if the teacher outlines how to resolve the situation in advance. In this case, the teacher sets up an explicit didactical contract (Brousseau, 1997) that problems will be given in such a way that students only have to supply certain known steps—the rest is set up for them. This creates a social partnership in which students need only ‘do their part’ by applying well-known methods in specific ways. They must divide the cubic polynomial by the given linear factor, factor the remaining quadratic, and then read off the solutions. Because it is possible to perform these steps without considering the meaning of the problem, students need not understand the situation or the purpose of actions they perform.

Instead of promising that students will not have to deal with finding intercepts when an equation does not factor nicely, the teacher could ask more probing questions to examine student conceptions and make them consider problematic issues. For example, he could ask “how many *x*-intercepts can a quadratic polynomial have?” and “how many *x*-intercepts can a cubic polynomial have?” These questions could lead to a *need for causality* if students want to know why it is that a cubic will always have at least one *x*-intercept but a quadratic may have none. They may be able to see that this happens from several graphs, but they are unlikely to understand why similar equations have different numbers of *x*-intercepts. This could lead to interesting discussion and possibly methods for bounding the number of intercepts. He could also ask students how they might ascertain whether a quadratic equation can be factored (over the integers). This would feed into a *need for computation* if students want to know when they should be able to factor easily and when they need to use the quadratic formula. Such considerations would force students to consider the problems they are working on carefully and to gauge how difficult finding the solution might be.

Episode 4.2

The teacher circulates around the room, talking to different students as they work on exponential growth and decay problems:

1. T: So, this is 1.63. What’s my growth factor? What’s my growth, what’s my rate, what’s my percent of increase?
2. S: I don’t know.
3. T: Look right up there. If b is 1 plus the percent, right?
4. S: Oh, so 1 plus 1.63?
5. T: I, no, I already did, that’s already added to 1, so what’s 1.63 minus 1?

(….)

6. T: What did you do for 21 [another problem]?

7. S2: [inaudible]
8. T: So you did, you did 1 minus 0.11.
9. S2: Yeah.
10. T: Right.
11. S2: We got 0.89, yeah.
12. T: Right, so that's 1 minus 0.11.
13. S2: Yeah.
14. T: Did you, did you do [pause] how did you [pause] did you do that on your calculator?
15. S3: Yeah.
16. T: Let me see. No, no, no, I want to see 1 minus 11 percent on the calculator.
17. S2: No, we didn't do it like that.
18. T: Oh, okay.
19. S3: We did 100 minus 11 to get 0.89.

(….)

20. T: So I did put some notes on [my website], but this is all it is right here. Growth factor is *b* is 1 plus the percent, decay is 1 minus the percent.

Students seem to be working on homework from the textbook, but the problem situations are not stated or discussed by the teacher in this episode. The teacher appears to be testing whether students can correctly apply a given rule: "b is 1 plus [or minus] the percent." His prescription, "I want to see 1 minus 11 percent on the calculator" provides only social necessity for a particular way of applying the method. The students who "did 100 minus 11 to get 0.89" seem to be modifying their answer to match the teacher's expectation without mentioning that they (presumably) process this answer as a percent, dividing by 100. The discussion focuses on results and what was put into the calculator, not on the meaning of "growth factor" in this context or why one would wish to apply this rule to determine the base of an exponential.

On the whiteboard, and presumably on the teacher's website, the equations "$b = 1 + \%$" and "$b = 1 - \%$" literally appear, despite being mathematically ungrammatical. This rule allows students to know what they must do before seeing a problem: figure out if the problem involves growth or decay, find "the percent" and either add it to one or subtract it from one. In general, it is not obvious what "the percent" refers to. The teacher's statement establishes a didactical contract that there will only be one quantity given as a percentage so that this quantity can be assumed to be, without considering its meaning in context, "the percent" (to be expressed as a decimal). Since students have no need to consider problems carefully, their activity will be problem-free. We also note that this seemingly straightforward rule does not allow students to answer the teacher's first questions easily.

To implement the necessity principle, several problems involving exponential growth and decay should be given before showing the general formula for them. Once students have some experience with particular situations, the teacher could introduce the general form $f(t) = Ab^t$ as the type of function underlying such problems (*need for connection and structure*). He could then ask where the values *A* and *b* come from in the problem situation. Students should be able to see that *A* is just the starting amount, whereas *b* comes from the rate of growth: more specifically, if a quantity is growing by *r*% per unit of time, then $= 1 + {}^{r}/_{100}$. Once they have made this realization, they can infer the rate of growth or decay from the form of an exponential function. A *need for communication* when comparing answers can lead students to agree on a particular form for the rate of growth (whether expressed as a decimal or percent).

HYPOTHESIZED CAUSES FOR THE OBSERVED PROBLEM-FREE ACTIVITY

Many of the teachers' actions in the above episodes, though seemingly intended to help the students, are likely to lead to problem-free activity in the classroom. Lack of content knowledge was sometimes a contributing factor to these actions: the teachers did not know why something had to be the way it is, or what alternative methods might work in the same situation. However, based on conversations with the teachers, we believe that the dominant cause for the actions we observed is the way these teachers view mathematics. They see what will be tested as "the mathematics" that they are supposed to teach. Because of this, they do not consider other options for solving problems. If the goal is for students to be able to use arithmetic sequences to solve certain problems on a test, then it does not matter whether there are easier ways to solve such problems or other ways of thinking about them. In their view, teachers are not supposed to stimulate students to think deeply about problems; rather, they are supposed to prepare students for testing and future courses by covering specific material. This view does not arise in a vacuum, but rather is shaped by external pressures. In recent years, the frequency and importance of mathematics testing have increased significantly. Teachers are expected to make sure most of their students perform adequately on these tests so that the school continues to receive funding. We hypothesize that, because the tests tend to cover a lot of material at a shallow level using specific types of problems, the curriculum teachers teach follows a similar pattern. This performance-driven attitude (where performance is measured by speed and accuracy rather than deep understanding) leads teachers to look for ways to speed up student learning, such as giving the students a fixed procedure before assigning problems. However, if one accepts the DNR premise that learning arises from solving problems that require students to go beyond their current knowledge, then such attempts become counterproductive.

Teachers may also believe that new methods should be introduced using the simplest possible problems, to minimize technical complications and extraneous features. However, simple problems are often solvable by simpler methods, in which case they will not create intellectual need for the method being taught. On the contrary, students may view the use of the new method as an arbitrary requirement imposed by the teacher.

We believe that these actions and attitudes are not restricted to the two teachers we observed; our impression is that mathematics classrooms across the country and at all age levels share many of these characteristics. Because of this, similar manifestations of problem-free activity are likely to occur. By attending to the intellectual need of their students and encouraging problem-laden, rather than problem-free, activity, teachers can help their students learn more. We have made particular suggestions for infusing the situations we observed with intellectual need.

IMPLICATIONS FOR TEACHING AND TEACHER TRAINING

We believe that problem-free activity occurs frequently in most mathematics classrooms. It deserves further scrutiny: teachers should learn to both identify and prevent such activity. Our observations and example scenarios suggest several guidelines for teaching with intellectual need.

1. Formulate long-term goals for instruction.

Teachers should have clear objectives for their overall class and each activity they give to students. Otherwise, it is highly unlikely that students will perceive any intellectual purpose behind their mathematical activity. Based on the curriculum they are given, teachers should formulate overarching questions that drive the material and select material that instantiates these questions and helps to achieve the goals of the course. They should also present tasks in an organized fashion so that the goals of each activity are clear to students.

2. Choose tasks carefully.

Our observations highlight the importance of the tasks assigned to students. In addition to furthering long-term goals, tasks should be clear, problematic, and reasonable—though they need not be real. Tasks should be understandable using students' existing knowledge, but they should require significant thinking to solve. At times it may be desirable to present more difficult tasks first, and then suggest that students consider how to formulate and solve a simpler version of the problem.

3. Emphasize the meaning of problems and their solutions.

Problems should set up meaningful situations rather than merely serving as triggers for some action. Teachers should check on how students understand problems, particularly when students encounter difficulty. They should not assume that students interpret problems in the same way they do. When problems are ambiguous, the teacher can prompt a discussion among students regarding which interpretation seems most reasonable. Moreover, teachers should emphasize that many mathematics problems have a physical (real-world) embodiment as well as geometric and algebraic representations. Key features of any of these three realizations can be expected to correspond to important features in the others, and these correspondences can be sought and exploited in problem-solving.

4. Allow students to explore their own methods of solving problems.

In general, problems or questions should be presented before procedures are introduced; in this way, the need for a procedure is supplied by a problem. Students should be encouraged to attack problems in their own ways, unfettered by procedural restrictions. The teacher can wait to introduce a standard algorithm until students have presented their own methods or encountered significant difficulties. In addition, problems that motivate standard algorithms should not be solvable by easier procedures—they should either require the algorithm being motivated or be awkward to solve by other means. When alternate solution methods arise, students should be encouraged to compare them based on intellectual criteria.

These guidelines have several implications for teacher training programs. As noted earlier, content knowledge alone is insufficient for good teaching. Teachers should not only learn how to correctly interpret and solve problems, but also they should explore common student interpretations and solution attempts for different types of problems. They should be taught to explain the meaning of a problem in relation to a student’s understanding of it. This can be quite difficult, particularly when students are not articulate in describing their interpretation. Teachers should be provided with coherent curricula that provide explicit links between the goals of instruction (both local and overarching) and the tasks that are presented to students. Teachers should understand these goals and how different tasks can achieve them. Teachers should also be trained to create and select tasks that will maximize learning opportunities for students by stimulating intellectual need. Finally, teachers should learn to value and pay attention to student thinking. Teachers and textbooks need not always be the sources of solution procedures; student ideas can drive much of the learning that occurs. To achieve these results, it is necessary to provide teachers with explicit examples of carefully-selected problem tasks that fit into a coherent unit. However, prevailing attitudes may need to change as well. Primary and secondary school teachers rarely encourage significant investment in understanding a particular mathematics problem. Instead, there is typically a strong expectation that students will quickly produce answers (though not always correct ones) to the problems they are given, leading to problem-free activity. In order to teach with intellectual need, teachers must set up a classroom environment in which making sense of a problem is more important than producing an answer. When students understand a problem thoroughly, the answers they offer are more likely to contain mathematical insight, even when those answers are not complete and correct. In addition, many student errors can be traced to the way students interpret problems, rather than simply their level of knowledge.

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